MATH 247 by Camila Sehnem

Eason Li

2024 S

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Lecture 1 - Monday, May 6

Recall that if $S_1, \dots, S_n$ are sets, then the **Cartesian Product** $S_1 \times \cdots \times S_n$, also denoted as $\prod_{i=1}^{n} S_i$, is the set

$$S_1 \times \cdots \times S_n = \{(x_1, \dots, x_n) \mid x_j \in S_j,\ j = 1, \dots, n\}$$

Definition 0.1: N-dimensional Euclidean Space, Vector

The **N-dimensional Euclidean Space** is the N-fold Cartesian product $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$. Element $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is called a **vector** is simply a point in $\mathbb{R}^n$. The numbers $x_1, \dots, x_n$ are called the coordinates.

Recall that $\mathbb{R}^n$ is a vector space over $\mathbb{R}$ with coordinate-wise operations: that is, if $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we have

$$\begin{aligned} x + y &= (x_1 + y_1, \dots, x_n + y_n) \\ \lambda x &= (\lambda x_1, \dots, \lambda x_n) \end{aligned}$$

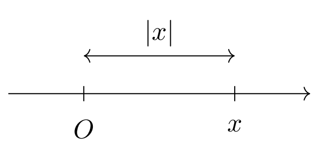
Definition 0.2: Zero Vector / Origin

The **zero vector**, or the **origin**, is the vector $\vec{0} = (0, \dots, 0)$.

1 The Euclidean Inner Product and Distance in $\mathbb{R}^n$

Example 1.1: Absolute Vector

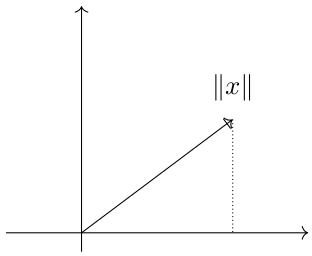
In $\mathbb{R}$, the distance of $x \in \mathbb{R}$ is from O in the **absolute vector**, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$.



For $x, y \in \mathbb{R}$, the distance of x and y is $|x - y|$.

Example 1.2: In $\mathbb{R}^2$

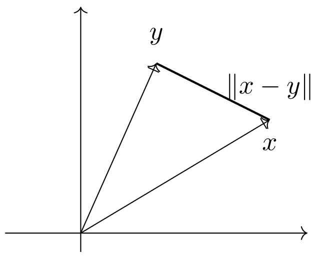
In $\mathbb{R}^2$, there is a natural notion of distance of a vector $x = (x_1, x_2)$ to 0.



$$\|x\| := \sqrt{x_1^2 + x_2^2}$$

For $x, y \in \mathbb{R}^2$, we can define the distance of x and y by

$$\|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$



We can find that $\|x - y\|^2 = \|x\|^2 + \|y\|^2$ if and only if the "dot product", $x \circ y = x_1 y_1 + x_2 y_2$ is zero, because $x \circ y = \|x\| \cdot \|y\| \cos\theta$ (follow from the law of cosine).

We extend this to $\mathbb{R}^n$

1.1 Standard Inner Product

Definition 1.1: Euclidean Inner Product (Dot Product)

The Euclidean inner product (or dot product) on $\mathbb{R}^N$ is the function

$$\begin{aligned} \circ : \mathbb{R}^N \times \mathbb{R}^N &\to \mathbb{R} \\ (x, y) &\to \sum_{i=1}^{N} x_i y_i \end{aligned}$$

Proposition 1.1

The dot product satisfies that for all $x, y \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$, the following holds:

1. $x \circ x \geq 0$
2. $x \circ x = 0$ if and only if $x = 0$
3. $x \circ y = y \circ x$
4. $x \circ (y + z) = x \circ y + x \circ z$
5. $(\lambda x) \circ y = \lambda(x \circ y)$

Result 1.1

Properties 3, 4 and 5 imply that $\circ$ is **bilinear**.

1.2 (Euclidean) Norm

Definition 1.2: Norm

For $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we define the (Euclidean) **norm** of x by

$$\|x\| = \sqrt{x \circ x} = \sqrt{\sum_{i=1}^{N} x_i^2}$$

Proposition 1.2

The function $\|\cdot\| : \mathbb{R}^N \to [0, \infty)$ satisfies

1. $\|x\| \geq 0$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\lambda x\| = |\lambda| \, \|x\|$

We would also like to show that this satisfies the **triangle inequality**:

$$\|x + y\| \leq \|x\| + \|y\| \quad \text{for all } x, y \in \mathbb{R}^N$$

For this we need the Cauchy-Schwartz inequality.

1.3 Cauchy-Schwartz Inequality

Theorem 1.1: Cauchy-Schwartz

For all $x, y \in \mathbb{R}^N$ we have

$$|x \circ y| \leq \|x\| \cdot \|y\|$$

Moreover, equality holds if and only if $x = ty$ or $y = tx$ for some $t \in \mathbb{R}$.

Proof. We may assume that both x and y are non-zero. For all $t \in \mathbb{R}$, we know that

$$(x - ty) \circ (x - ty) \geq 0$$

then we have

$$p(t) = x \circ x - 2t(x \circ y) + t^2(y \circ y) \geq 0$$

Notice that this is a quadratic function of t, which implies that $p(t)$ has at most one root, thus

$$\Delta = \left[2(x \circ y)^2\right] - 4(x \circ x)(y \circ y) \leq 0$$

and the remaining follows naturally. $\square$

Corollary 1.1: Triangle Inequality

For all $x, y \in \mathbb{R}^N$ we have

$$\|x + y\| \leq \|x\| + \|y\|$$

Proof. We simply have

$$\begin{aligned} \|x + y\|^2 &= (x + y) \circ (x + y) \\ &= \|x\|^2 + \|y\|^2 + 2(x \circ y) \\ &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$

thus completing the proof. $\square$

Lecture 2 - Wednesday, May 8

Theorem 1.2: Properties of the Euclidean Norm

The Euclidean norm $\|\cdot\| : \mathbb{R}^N \to [0, \infty)$ satisfies the following for all $x, y \in \mathbb{R}^N$ and $\lambda \in \mathbb{R}$:

1. Proposition 1.2
2. Triangle inequality
$$\|x + y\| \leq \|x\| + \|y\|$$
3. Reversed triangle inequality
$$|\, \|x\| - \|y\| \,| \leq \|x - y\|$$

Proof. exercise. $\square$

Definition 1.3: Distance

For $x, y \in \mathbb{R}^N$, define the distance of x and y by

$$d(x, y) := \|x - y\|$$

Notice that for all $z \in \mathbb{R}^N$,

$$d(x, y) \leq d(x, z) + d(z, y)$$

which is a direct consequence of the Triangle Inequality 1.1.

2 Angles between Vectors in $\mathbb{R}^N$

In $\mathbb{R}^2$, we know that $x \circ y = \|x\| \, \|y\| \cos\theta$, where θ is the angle between x and y.

In $\mathbb{R}^N$, Cauchy-Schwartz inequality 1.1 implies that for $x, y \neq 0$, then

$$\frac{x \circ y}{\|x\| \, \|y\|} \in [-1, 1]$$

we can find a unique $\theta \in [0, \pi]$ such that

$$\cos\theta = \frac{x \circ y}{\|x\| \, \|y\|}$$

Definition 2.1: Angle between x and y

We define the **angle between** x **and** y as θ.

2.1 Orthogonal

Definition 2.2: Orthogonal

We say x and y are **orthogonal** if $\theta = \pi/2$.

3 Topology on $\mathbb{R}^N$ - Open Sets and Closed Sets

In topology, we study the notion of **closeness** (limits, convergence, continuity, etc.) through the collection of open sets / closed sets.

Definition 3.1: Open Ball and Closed Ball

The **open ball** in $\mathbb{R}^N$ of radius $r > 0$ centered at $x \in \mathbb{R}^N$ is the set

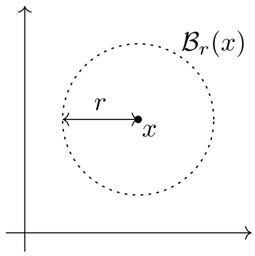
$$\mathcal{B}_r(x) = \{y \in \mathbb{R}^N \ : \ \|x - y\| < r\}$$

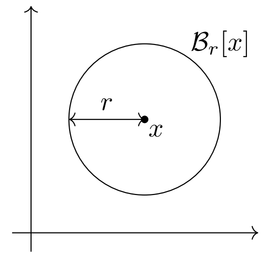
Remark: the other notation is $\mathcal{B}(x, r)$.
The **closed ball** in $\mathbb{R}^N$ of radius $r > 0$ centered at $x \in \mathbb{R}^N$ is the set

$$\mathcal{B}_r[x] = \{y \in \mathbb{R}^N \ : \ \|x - y\| \leq r\}$$

Example 3.1

1. In $\mathbb{R}$, $\mathcal{B}_r(x)$ is the open interval $(x - r, x + r)$. Similarly, $\mathcal{B}_r[x]$ is the closed interval $[x - r, x + r]$.
2. In $\mathbb{R}^2$, we have





Definition 3.2: Open Set and Closed Set

1. We say that $U \subseteq \mathbb{R}^N$ is **open** if for all $x \in U$, there exists $\varepsilon > 0$ (depending on x) such that $\mathcal{B}_\varepsilon(x) \subseteq U$.

2. We say that $F \subseteq \mathbb{R}^N$ is **closed** if its complement,

$$F^c = \{y \in \mathbb{R}^N \ : \ y \notin F\},$$

is open.

Result 3.1: "Clopen"

Notice that $\varnothing$ and $\mathbb{R}^N$ are open; and they are also closed. They are known as **clopen**.

Proposition 3.1: Open Balls are Open, and Vice Versa

1. The open ball $\mathcal{B}_r(x)$ is open.

2. The closed ball $\mathcal{B}_r[x]$ is closed.

Proof. The proof consists of two parts:
(Part 1):
Let $y \in \mathcal{B}_r(x)$, we want to find $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(y) \subseteq \mathcal{B}_r(x)$. We know that for $z \in \mathbb{R}^N$

$$d(x, z) \ \leq \ d(x, y) + d(y, z)$$

hence we can take $\varepsilon = r - d(x, y)$, then $\varepsilon > 0$ and $\mathcal{B}_\varepsilon(y) \subseteq \mathcal{B}_r(x)$.
(Part 2):
Use the Reversed Triangle Inequality:

$$|\,\|x - z\|\,| = \|x - y + y - z\| \geq |\,\|x - y\| - \|z - y\|\,|$$

We want to show that

$$\mathcal{B}_r[x]^c = \{y \in \mathbb{R}^N \ : \ \|y - x\| > r\}$$

is open. Choose y such that $\|y - x\| > r$. Let $\varepsilon = \|x - y\| - r$, so $\varepsilon > 0$. Also let $z \in \mathcal{B}_\varepsilon(y)$, then we have

$\|z-y\| < \varepsilon$, which implies that $-\|z-y\| > -\varepsilon = r - \|x-y\|$. Therefore,

$$\begin{aligned} \|x-z\| &\geq |\, \|x-y\| - \|y-z\| \,| \\ &= |\, \|x-y\| - \|z-y\| \,| \\ &> \|x-y\| + r - \|x-y\| \\ &= r \end{aligned}$$

Hence $z \in \mathcal{B}_r[x]^c$ is needed wich means that $\mathcal{B}_\varepsilon(y) \subseteq \mathcal{B}_r[x]^c$. □

3.1 Permanence Properties of Open Sets

Theorem 3.1: Permanence Properties of Open Sets

1. The union of an arbitrary collection of open sets is open.
   Precisely, if Λ are indices and $\{E_\alpha \mid \alpha \in \Lambda\}$ are open sets, then

$$E \equiv \bigcup_{\alpha \in \Lambda} E_\alpha$$

   is open.

2. The intersect of a *finite* collection of open sets is open.

Proof. 1. Let $x \in E$, then there exists $\alpha \in \Lambda$ such that $x \in E_\alpha$. Since E_α is open, then there exists some $\varepsilon > 0$ such that

$$\mathcal{B}_\varepsilon(x) \subseteq E_\alpha \subseteq \bigcup_{\alpha \in \Lambda} E_\alpha = E$$

which implies that E is also open

2. Let $E_1, E_2, \ldots, E_m$ be open sets in $\mathbb{R}^N$ and we let $E \equiv \bigcap_{i=1}^m E_i$. Let $x \in E$. For $i = 1, \ldots, m$, we can find $\varepsilon_i > 0$ such that $\mathcal{B}_\varepsilon(x) \subseteq E_i$. So we can set $\varepsilon \equiv \min\{\varepsilon_i \ : \ i = 1, \ldots, m\}$. Then

$$\mathcal{B}_\varepsilon(x) \subseteq \bigcap_{i=1}^{m} E_i = E$$

giving that E is open.

□

Lecture 3 - Friday, May 10

Example 3.2

The intersection of an infinite collection of open sets need not to be open, Consider that for all $m \geq 1$. take $E_m \equiv \mathcal{B}_{1/m}(n)$, then E_m is open, but the intersect is a single point n, which is indeed closed.

3.2 De Morgan's Law

Theorem 3.2: De Morgan's Law

Let $\{E_\alpha \;:\; \alpha \in \Lambda\}$ be a collection of subsets of a set A, then

$$\left(\bigcup_{\alpha\in\Lambda} E_\alpha\right)^c = \bigcap_{\alpha\in\Lambda} E_\alpha^c$$
$$\left(\bigcap_{\alpha\in\Lambda} E_\alpha\right)^c = \bigcup_{\alpha\in\Lambda} E_\alpha^c$$

Corollary 3.1: Properties of Closed Sets

1. The intersection of an arbitrary collection of closed sets is closed
2. The union of a finite collection of closed sets is closed

Proof. This follows the De Morgan's Law 3.2. □

Example 3.3

The sphere

$$\partial\mathcal{B}_r(x) = \{y \in \mathbb{R}^N \;:\; \|y - x\| = r\}$$

is closed because

$$\partial\mathcal{B}_r(x) = \mathcal{B}_r[x] \cap \mathcal{B}_r(x)^c$$

Example 3.4

The union of an infinite collection of closed sets need not be closed: Take $F_m = \{1/m\}$ (i.e. $(1/m, \ldots, 1/m) \in \mathbb{R}^N$), then F_m is closed,
Exercise: Show that $\bigcup_{m=1}^{\infty} F_m$ is not closed.

Proof. To show that $\bigcup_{m=1}^{\infty} F_m$ is not closed, it suffices to show that the complement is not open. Consider the point $\mathcal{O} = \{0\}$, we can easily find that we are not able to construct an open ball that is contained in the complement, thus completing the proof. □

4 Sets that are neither closed not open

Discovery 4.1

In general, an arbitrary subset S of $\mathbb{R}^N$ need not be closed nor open.

Example 4.1

In $\mathbb{R}$, consider $(a, b]$.

Example 4.2

Let

$$S \equiv \{(x, y, z) \in \mathbb{R}^3 \ : \ y^2 + z^2 = 1, x > 0\}$$

then S is neither closed nor open.

Proof. 1. (not open)
Take $p = (1, 0, 1) \in S$, then for $\varepsilon > 0$, we claim that $\mathcal{B}_\varepsilon(p) \cap S^c \neq \varnothing$ (i.e. there are points in the open ball around p but not in S). We can simple set the point to be $q = (1, 0, 1 + \varepsilon/2)$.

2. (not closed)
Take $p = (0, 0, 1) \in S^c$, given that $\varepsilon > 0$, we want to show that S^c is not open. Take $q = (\varepsilon/2, 0, 1)$, then $q \in S$ and $q \in \mathcal{B}_\varepsilon(p)$, so $\mathcal{B}_\varepsilon(p) \cap S \neq \varnothing \ \Rightarrow \ \mathcal{B}_\varepsilon(p) \not\subseteq S^c$.

□

4.1 Cluster Point

Definition 4.1: Cluster Point

1. A point $p \in \mathbb{R}^N$ is called a **cluster point** (or accumulation point) of S if for every $\varepsilon > 0$, we have

$$(\mathcal{B}_\varepsilon(p) \backslash \{p\}) \cap S \neq \varnothing$$

Equivalently, for every open set U with $p \in U$, there exists $x \in S \cap U$ and $x \neq p$.

2. We denote by S' the set of all cluster points of S.

Example 4.3: Every $p \in \mathbb{R}^N$ is a cluster point of $\mathbb{Q}^N$

Every $p \in \mathbb{R}^N$ is a cluster point of $\mathbb{Q}^N = \{(q_1, \ldots, q_N) \in \mathbb{R}^N \ : \ q_i \in \mathbb{Q}, \ i = 1, \ldots, N\}$.

Proof. To see this, let $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$ and $\varepsilon > 0$. By density of $\mathbb{Q}$ in $\mathbb{R}$, for each $i = 1, \ldots, N$, we can find $c_i \in \mathbb{Q}$, $c_i \neq p_i$ such that $|p_i - c_i| < \varepsilon/\sqrt{N}$, set $c = (c_1, \ldots, c_N) \in \mathbb{Q}^N$, then

$$\|p - c\| = \sqrt{\sum_{i=1}^{N} (p_i - c_i)^2} < \varepsilon$$

and $p \neq c$. Hence $c \in (\mathcal{B}_\varepsilon(p) \backslash \{p\}) \cap \mathbb{Q}^N$ is needed. □

Example 4.4

Let S be a finite set,

$$S = \{x_1, \ldots, x_N\} \in \mathbb{R}^N$$

then S has no cluster point.

Proof. To see this, take $p \in \mathbb{R}^N$ and $\varepsilon > 0$ with

$$\varepsilon < \min\{\|p - x\| \ : \ x \in S, x \neq p\}$$

□

4.2 Characterization of Closed Sets

Theorem 4.1: Characterization of Closed Sets

Let $F \subseteq \mathbb{R}^N$, TFAE

1. F is closed
2. $F' \subseteq F$

Proof.

1. $(1 \Rightarrow 2)$
   Suppose F is closed. Let $p \in F^c$, we have to show that $p \notin F'$. Since F is closed, F^c is open, hence there exists $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(p) \subseteq F^c$. In particular,

   $$\mathcal{B}_\varepsilon(p) \cap F = \varnothing$$

   giving that $p \notin F'$, we have $F' \subseteq F$.

2. $(2 \Rightarrow 1)$
   Suppose $F' \subseteq F$, we will show that F^c is open. Take $p \in F^c$, then $p \notin F'$, so there exists $\varepsilon > 0$ such that

   $$(\mathcal{B}_\varepsilon(p) \backslash \{p\}) \cap F = \varnothing$$

   Thus $\mathcal{B}_\varepsilon(p) \cap F = \varnothing$. Since $p \in F^c$, so $\mathcal{B}_\varepsilon(p) \subseteq F^c$ and thus F^c is open.

□

4.3 Closure

Definition 4.2: Closure

Let $S \subseteq \mathbb{R}^N$, define the **closure** of S by $\overline{S} = S \cup S'$.

Proposition 4.1

Let $S \subseteq \mathbb{R}^N$. Then

$$S' = \overline{S}'$$

In particular, we have $\overline{S}$ is closed.

Corollary 4.1

$\overline{S}$ is the smallest closed set contans S. i.e. if $S \subseteq F$ and F is closed, then $\overline{S} \subseteq F$.

$$\overline{S} = \bigcap_{\substack{F \supseteq S \\ F \text{ open}}} F$$

Definition 4.3: Boundary and Interior

Let $S \subseteq \mathbb{R}^N$,

1. We say that a point $p \in \mathbb{R}^N$ is a **boundary point** of S if for every $\varepsilon > 0$, we have

$$\mathcal{B}_\varepsilon(p) \cap S \neq \varnothing \qquad \& \qquad \mathcal{B}_\varepsilon(p) \cap S^c \neq \varnothing$$

   The **boundary**, ∂S, is the set of all boundary points of S.

2. We say that a point $p \in \mathbb{R}^N$ is an **interior point** of S if there exists $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(p) \subseteq S$. The **interior** of S, denoted by S°, is the set of all interior points of S.

Result 4.1

We have

$$S^\circ \subseteq S \subseteq \overline{S}$$

Example 4.5

Let $S = (0,1] \cup \{2\}$, we have

$$\begin{aligned} \partial S &= \{0, 1, 2\} \\ S' &= [0,1] \\ S^\circ &= (0,1) \\ \overline{S} &= [0,1] \cup \{2\} \end{aligned}$$

Proposition 4.2

Let $x \in \mathbb{R}^N$ and $r > 0$, then

1. $\partial\mathcal{B}_r(x) = \partial\mathcal{B}_r[x] = \{y \in \mathbb{R}^N \ : \ \|y - x\| = r\}$
2. $\overline{\mathcal{B}_r(x)} = \mathcal{B}_r[x]$

Proof. 1. Let $y \in \mathbb{R}^N$ with $\|y - x\| = r$. It suffices to show that for all $\varepsilon > 0$,

$$\mathcal{B}_\varepsilon(y) \cap \mathcal{B}_r(x) \neq \varnothing \quad \& \quad \mathcal{B}_\varepsilon(y) \cap \mathcal{B}_r[x]^c \neq \varnothing$$

since $\mathcal{B}_r(x)$ and $\mathcal{B}_r[x]^c$ are open. Let $\lambda > 0$, so we have

$$\|\lambda(y - x)\| = \lambda \|y - x\| = \lambda r$$

Set $z_\lambda = x + \lambda(y - x)$. Notice that if $\lambda < 1$, then $z_\lambda \in \mathcal{B}_r(x)$, and if $\lambda > 1$, then $z_\lambda \in \mathcal{B}_r[x]^c$. Take $0 < \lambda < 1$ with $1 - \lambda < \varepsilon/r$, then $z_\lambda \in \mathcal{B}_r(x)$ and

$$\begin{aligned} \|z_\lambda - y\| &= \|x + \lambda(y - x) - y\| \\ &= (1 - \lambda) \|y - x\| \\ &< \frac{\varepsilon}{r} \cdot r = \varepsilon \end{aligned}$$

To get $z_\lambda \in \mathcal{B}_\varepsilon(y) \cap \mathcal{B}_r[x]^c$, take $\lambda > 0$ with $\lambda - 1 < \varepsilon/r$, then $z_\lambda \in \mathcal{B}_r[x]^c$ and is above $z_\lambda \in \mathcal{B}_\varepsilon(y)$.

2. We know that

$$\overline{\mathcal{B}_r(x)} = \mathcal{B}_r(x) \cup \mathcal{B}_r(x)'$$

If $p \in \mathcal{B}_r[x]^c$, then $p \notin \mathcal{B}_r(x)'$, so

$$\overline{\mathcal{B}_r(x)} \subseteq \mathcal{B}_r[x]$$

By part a), if $p \in \mathbb{R}^N$ and $\|p - x\| = r$, then $p \in \partial\mathcal{B}_r(x)$ and hence $p \in \mathcal{B}_r(x)$, thus

$$\mathcal{B}_r[x] \subseteq \overline{\mathcal{B}_r(x)}$$

□

Proposition 4.3

Let $S \subseteq \mathbb{R}^N$, then

1. S° is open, and

$$S^\circ = \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$$

2. $S^\circ = S \backslash \partial S$

Proof. 1. Let $x \in S^\circ$, since x is an interior point, so we can find $\varepsilon_x > 0$ such that

$$\mathcal{B}_{\varepsilon_x}(x) \subseteq S$$

If $y \in \mathcal{B}_{\varepsilon_x}(x)$, then there exists $\delta > 0$ such that

$$\mathcal{B}_\delta(y) \subseteq \mathcal{B}_{\varepsilon_x}(x) \subseteq S$$

So y is also an interior point. This gives that

$$\mathcal{B}_{\varepsilon_x}(x) \subseteq S^\circ$$

This shows that S° is open, and

$$S^\circ = \bigcup_{x \in S^\circ} \mathcal{B}_{\varepsilon_x}(x) \subseteq \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$$

Now let $U \subseteq S$, U open and let $x \in U$. Since U is open, there exists $\varepsilon > 0$ such that

$$\mathcal{B}_\varepsilon(x) \subseteq U \subseteq S$$

suggesting that $x \in S^\circ$, hence completes the proof.

2. Let $x \in S^\circ$, we want to show that $x \notin \partial S$. We know there exists $\varepsilon > 0$ such that

$$\mathcal{B}_\varepsilon(x) \subseteq S$$

hence we have

$$\mathcal{B}_\varepsilon(x) \cap S^c = \varnothing \quad \implies \quad S^\circ \subseteq S \backslash \partial S$$

On the other hand, let $x \in S \backslash \partial S$, hence we can find $\varepsilon > 0$ such that

$$\mathcal{B}_\varepsilon(x) \cap S^c = \varnothing \quad \implies \quad x \in S^\circ$$

□

Lecture 5 - Wednesday, May 15

Discovery 4.2

S° is the largest open set contained in S.

4.4 $\mathbb{R}^N$ is the Disjoint Union

Theorem 4.2

Let $S \subseteq \mathbb{R}^N$, then $\mathbb{R}^N$ is the disjoint union

$$\mathbb{R}^N = S^\circ \sqcup \partial S \sqcup (S^c)^\circ$$

Remark: The symbol $\sqcup$ implies that this is a disjoint union.

Proof. Clearly $S^\circ \cap (S^c)^\circ = \varnothing$ since $S^\circ \subseteq S$ and $S^{c\circ} \subseteq S^c$, and if $p \in S^\circ \cup (S^c)^\circ$, then $p \notin \partial S$, thus the above union is disjoint. To see that $\mathbb{R}^N = S^\circ \cup \partial S \cup (S^c)^\circ$, let $x \in \mathbb{R}^N$, if $x \in S^\circ \cup (S^c)^\circ$, we are done. Otherwise given $\varepsilon > 0$, we have $\mathcal{B}_\varepsilon(x) \cap S^c \neq \varnothing$ because $x \notin S^\circ$ and $\mathcal{B}_\varepsilon(x) \cap S \neq \varnothing$ because $x \notin (S^c)^\circ$. Since ε is arbitrary, thus we have $x \in \partial S$. ☐

Corollary 4.2

For any $S \subseteq \mathbb{R}^N$, we have

$$\overline{S} = S \cup \partial S$$

Proof. Exercise. ☐

5 Compactness

Compactness is an important concept in topology especially in connection with continuity.

Definition 5.1: Open Cover, Compact

1. Let $S \subseteq \mathbb{R}^N$. An **open cover** of S is a collection, $g = \{g_\alpha\}_{\alpha \in \Lambda}$, of open subsets of $\mathbb{R}^N$ that covers S. i.e.

$$S \subseteq \bigcup_{\alpha \in \Lambda} g_\alpha$$

2. We say that $K \subseteq \mathbb{R}^N$ is **compact** if every open cover $g = \{g_\alpha\}_{\alpha \in \Lambda}$ of K admits a finite subcover. i.e. there exists a finite subcollection $g' = \{g_{\alpha_i} : i = 1, \ldots, n\}$ of sets from g such that

$$K \subseteq \bigcup_{i=1}^{n} g_{\alpha_i}$$

Example 5.1: Finite Sets Are Compact

If $S = \{x_1, \ldots, x_n\}$ is finite, then S is compact.

Proof. Let $g = \{g_\alpha\}_{\alpha \in \Lambda}$ be an open cover of S. Since $S \subseteq \bigcup_{\alpha \in \Lambda} g_\alpha$, for each $i = 1, \ldots, n$, we can find $\alpha_i \in \Lambda$ such that $x_i \in g_{\alpha_i}$. Set $g' = \{g_{\alpha_i} : i = 1, \ldots, n\}$, then g' is a finite collection of sets from g that cover S, thus S is indeed compact. ☐

Example 5.2: Open Balls Are Not Compact

Let $r > 0$, $x \in \mathbb{R}^N$, then $\mathcal{B}_r(x)$ is not compact.

Proof. We need to exhibit an open cover $g = \{g_\alpha\}_{\alpha\in\Lambda}$ that admits no finite subcover. Let $k \geq 0$ be such that $1/k < r$. For each $m \geq k$, we set $g_m = \mathcal{B}_{r-1/m}(x)$. Then each g_m is open and we set $g = \{g_m\}_{m\geq k}$. Then g is a open cover of $\mathcal{B}_r(x)$. We claim that g admits no finite subcover. SFAC that $g' = \{g_{m_i} : i = 1, \ldots, l\}$ is a finite subover for $\mathcal{B}_r(x)$. Let j be such that $m_j = \max\{m_i : i = 1, \ldots, l\}$. Then

$$\mathcal{B}_r(x) \subseteq g_{m_j} = \mathcal{B}_{r-1/m_j}(x)$$

which is a contradiction because if $u \in \mathbb{R}^N$, $\|u\| = 1$, and let $r - 1/m_j < q < r$, then $z = x + qu$, $z \in \mathcal{B}_r(x)$, but $z \notin \mathcal{B}_{r-1/m_j}(x)$. □

Proposition 5.1

Suppose $K \subseteq \mathbb{R}^N$ is compact and $F \subseteq K$ is closed, then F is compact.

Proof. Let $g = \{g_\alpha\}_{\alpha\in\Lambda}$ be an arbitrary open cover of F, then

$$K \subseteq F \cup F^c \subseteq \left(\bigcup_{\alpha\in\Lambda} g_\alpha\right) \cup F^c$$

so $\overline{g} = g \cup \{F^c\}$ is an open cover of K because F^c is open. Since K is compact, $\overline{g}$ admits a finite subcover $\overline{g}' = \{g_{\alpha_i} : i = 1, \ldots, n\}$. Now

$$\begin{aligned} F = F \cap K &\subseteq F \cap \left(\bigcup_{i=1}^n g_{\alpha_i}\right) \\ &= \bigcup_{i=1}^n F \cap g_{\alpha_i} \\ &\subseteq \bigcup_{g\in\overline{g}', g\neq F^c} g \end{aligned}$$

Setting $g' = \overline{g}' \backslash \{F^c\}$, we see that g' is a finite subcover of F containing of sets from g. □

Definition 5.2: Bounded

We will say that a set $S \subseteq \mathbb{R}^N$ is **bounded** if there exists $m \geq 1$ such that

$$S \subseteq \mathcal{B}_m[0]$$

Theorem 5.1

Suppose $K \subseteq \mathbb{R}^N$ is compact, then K is closed and bounded.

Proof. Suppose K is compact

1. *Bounded*:
   For each $m \geq 1$, let $g_m = \mathcal{B}_m(0)$, then $g_m \subseteq g_{m+1}$, and g_m is open. Let $g = \{g_m\}_{m\geq 1}$, then g is now an open cover of K. By compactness of K, g admits a fintie subcover $g' = \{g_{m_i} : i = 1, \dots, l\}$. Let j be such that $m_j = \max\{m_i : i = 1, \dots, l\}$. Then $K \subseteq g_{m_j} \subseteq \mathcal{B}_{m_j}[0]$.

2. *Closed*:
   For each $x \in K^c$, we need to find $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(x) \subseteq K^c$. For each $y \in K$, we set $\varepsilon_y = \|x - y\|/2$, then $\varepsilon_y > 0$ because $x \notin K$. By the reverse triangle inequality, we have

   $$\mathcal{B}_{\varepsilon_y}(x) \cap \mathcal{B}_{\varepsilon_y}(y) = \varnothing$$

   For each $y \in K$, we set $g_y = \mathcal{B}_{\varepsilon_y}(y)$ and let

   $$g := \{g_y : y \in K\}$$

   Then g is an open cover of K. By compactness, we can find a finite subcover from g, say $g' = \{g_{y_j} : j = 1, \dots, n\}$. Let $\varepsilon = \min\{\varepsilon_{y_j} : j = 1, \dots, n\}$.

□

Lecture 6 - Friday, May 17

Discovery 5.1

If $F \subseteq \mathbb{R}^N$ is closed and $K \subseteq \mathbb{R}^N$ is compact (so it's also closed and bounded), then $F \cap K$ is compact, since $F \cap K$ is closed (3.1) and $F \cap K \subseteq K$ (5.1).

Theorem 5.2

If $E \subseteq K$ is an infinite set and K is compact, then E has a cluster point in K.

Proof. SFAC that E has no cluster point in K. Since $E \subseteq K$, by A01-Q4, we have

$$E' \subseteq K' \subseteq K$$

because K is closed. Thus we must have $E' = \varnothing$. Then E is closed since $E' = \varnothing \subseteq E$. It follows that E is compact (5.1). Now if $p \in E$, it is clear that $p \notin E'$, so we get $\varepsilon_p > 0$ such that

$$\mathcal{B}_{\varepsilon_p}(p) \cap E = \{p\}$$

Then the open cover $\{\mathcal{B}_{\varepsilon_p}(p) : p \in E\}$ admits no finite subcover because E is infinite. □

5.1 Heine-Boul Theorem

We wish to prove the converse, that is, we want to show that if $K \sqsubseteq \mathbb{R}^N$ is closed and bounded, then K is compact.

Theorem 5.3: Nested Interval Principle

Recall the nested interval principle:

If $I_m = [a_m, b_m] \subseteq \mathbb{R}$ is a nested sequence of closed and bounded intervals in $\mathbb{R}$, then

$$\bigcap_{m=1}^{\infty} I_m \neq \varnothing$$

i.e. $I_m \supseteq I_{m+1} \supseteq \cdots$ for all m. Moreover, if $\lim_m (b_m - a_m) = 0$, then

$$\bigcap_{m=1}^{\infty} = \{z\}$$

is a single point.

Definition 5.3: N-cell

For each $j = 1, \ldots, N$, let $a_j, b_j \in \mathbb{R}$ with $a_j < b_j$. We call the Cartesian product

$$I = [a_1, b_1] \times \cdots \times [a_N, b_N]$$

an N-cell.

Theorem 5.4

Let $I_1 \supseteq I_2 \supseteq \cdots$ be an increasing sequence of N-cells, then

$$\bigcap_{m=1}^{\infty} I_m \neq \varnothing$$

Moreover, if $\lim_m \|b_m - a_m\| = 0$, then

$$\bigcap_{m=1}^{\infty} I_m = \{z\}$$

is a single point, where here $a_m, b_m \in \mathbb{R}^N$ and $I_m = [a_{m,1}, b_{m,1}] \times \cdots \times [a_{m,N}, b_{m,N}]$.

Proof. Since $I_m \supseteq I_{m+1}$, we have

$$[a_{m,j}, b_{m,j}] \supseteq [a_{m+1,j}, b_{m+1,j}]$$

By nested interval principle in $\mathbb{R}$, there exists

$$z_j \in \bigcap_{m=1}^{\infty} [a_{m,j}, b_{m,j}] \qquad j = 1, \ldots, N$$

We set $z = (z_1, \ldots, z_N)$, then $z \in \bigcap_{m=1}^{\infty} I_m$. If $\lim_{m\to\infty} \|b_m - a_m\| = 0$, then since $(b_{m,j} - a_{m,j}) \leq$

$\|b_m - a_m\|$, we deduce that $\lim_{n\to\infty}(b_{m,j} - a_{m,j}) = 0$. Hence

$$\bigcap_{m=1}^{\infty} [a_{m,j}, b_{m,j}] = \{z_j\}$$

Then

$$\bigcap_{m=1}^{\infty} I_m = \{z\}$$

□

5.2 N-cell is Compact

Theorem 5.5

Let $I = [a_1, b_1] \times \cdots [a_N, b_N]$ be an N-cell, then I is compact.

Theorem 5.6: Heine-Boul Theorem

Let $K \subseteq \mathbb{R}^N$, then TFAE

1. K is compact,
2. K is closed and bounded.

Proof. We have shown that compact implies closed and bounded (5.1). Therefore it suffices to show the other direction: Suppose K is closed and bounded. Since it is bounded, we find that there exists some $M > 0$ such that $K \subseteq \mathcal{B}_M[0]$. Then if $x \in K$, we have $|x_j| \leq \|x_j\| \leq M$ and so K is contained in the N-cell

$$I_M = \underbrace{[-M, M] \times \cdots \times [-M, M]}_{N \text{ terms}}$$

By the previous theorem 5.5, I_m is compact, and because $K \subseteq I_M$ and K is closed, thus K is compact (5.1). □

Lecture 7 - Tuesday, May 21

LMAO Camila didn't show up to the class today.

Lecture 8 - Wednesday, May 22

Proof. This is the proof of Theorem (5.5).
Let $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$. Set

$$\delta = \|b - a\| = \sqrt{\sum_{i=1}^{N}(b_i - a_i)^2}$$

Notice that if $x, y \in I$, then $\|x - y\| \leq \delta$. SFAC that I is not compact, then there exists an open cover $g = \{g_\alpha\}_{\alpha\in\Lambda}$ for I that admits no finite subcover.

1. **Step 1**:
   For each $j = 1, \ldots, N$, let $c_j = \frac{a_j+b_j}{2}$. Then the intervals $[a_j, c_j], [c_j, b_j]$ gives 2^N N-cells,

   $$J_1 = \{I_{1,l} : l = 1, \ldots, 2^N\}$$

   such that $I = \bigcup_{l=1}^{2^N} I_{1,l}$, where each N-cell $I_{1,l}$ is the Cartesian product

   $$[d_1, e_1] \times \cdots \times [d_N, e_N]$$

   with

   $$[d_j, e_j] \in \{[a_j, c_j], [c_j, b_j]\}$$

   It follows that there is some $l \in \{1, \ldots, 2^N\}$ such that the N-cell $I_{1,l}$ cannot be covered by a finite collection of sets from g. Let I_1 be such an N-cell. Notice that

   (a) $I \supseteq I_1$

   (b) I_1 cannot be covered by a finite collection of sets from g

   (c) Let $a_1 = (a_{11}, \ldots, a_{1N})$ and $b_1 = (b_{11}, \ldots, b_{1N})$ be such that

   $$I_1 = [a_{11}, b_{11}] \times \cdots \times [a_{1N}, b_{1N}]$$

   then if $x, y \in I_1$,

   $$\|x - y\| \leq \|b_1 - a_1\| = \sqrt{\sum_{i=1}^{N} (b_{1i} - a_{1i})^2} = \frac{\delta}{2}$$

2. **Step 2**:
   Induction. Suppose $n \geq 1$ is fixed and $I \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$ are N-cells where each I_l cannot be covered by a finite collection of sets from g, and if $x, y \in I_l$, we have $\|x - y\| \leq \delta/2^l$. Repeat the argument in step 1 to get an N-cell $I_{n+1} \subseteq I_n$ that cannot be covered by a finite collection of sets from g and $x, y \in I_{l+1}$, then $\|x - y\| \leq \delta/2^{n+1}$. We have proved the existence of a sequence $I, I_1, I_2, \ldots$ with the following properties:

   (a) $I \supseteq I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$

   (b) Each I_n cannot be covered by a finite collection of sets from g

   (c) If $x, y \in I_n$, then $\|x - y\| \leq \delta/2^n$

   By Theorem 5.3 we can find $z \in \bigcap_{n=1}^{\infty} I_n$. Since $z \in I \subseteq \bigcup_{\alpha \in \Lambda} g_\alpha$, there exists some $\beta \in \Lambda$ such that $z \in g_\beta$. Because g_β is open, there exists $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(z) \subseteq g_\beta$. Let n be such that $\delta/2^n < \varepsilon$. We know that $z \in I_n$ and if $y \in I_n$, we have

   $$\|y - z\| \leq \frac{\delta}{2^n} < \varepsilon$$

   giving that $y \in \mathcal{B}_\varepsilon(z)$. This shows that

   $$I_n \subseteq \mathcal{B}_\varepsilon(z) \subseteq g_\beta$$

   which is a contradiction because I_n can be covered by the singleton $\{g_\beta\} \in g$.

□

6 Connected Sets

Intuitively, a set $S \subseteq \mathbb{R}^N$ is **connected** if any two points $x, y \in S$ can be connected by a continuous path that is completely contained in S.

We define connected sets using topology.

Definition 6.1: Disconnection and Connection

Let $S \subseteq \mathbb{R}^N$ be a set. We say that a pair of open set $\{U, V\} \in \mathbb{R}^N$ is a **disconnection** for S if

1. $S \subseteq U \cup V$
2. $S \cap U \neq \varnothing$ and $S \cap V \neq \varnothing$
3. $S \cap U \cap V = \varnothing$

If a disconnection exists, we say that S is **disconnected**. Otherwise we say S is **connected**.

Example 6.1

$\mathbb{Z}$ is not connected, set $U = (-\infty, 1/2)$ and $V = (1/2, +\infty)$.
$\mathbb{Q}$ is not connected, set $U = (-\infty, \sqrt{2})$ and $V = (\sqrt{2}, +\infty)$

6.1 Interval is Connected

Theorem 6.1

The interval $[0, 1]$ is connected.

Lecture 9 - Friday, May 24

Proof. SFAC that $\{U, V\}$ is a disconnection. WLOG we may assume $0 \in U$. Since U is open, there exists some $\varepsilon_0 > 0$ such that $(-\varepsilon_0, \varepsilon_0) \subseteq U$. We may assume $\varepsilon_0 < 1$. Then $[0, \varepsilon_0) \subseteq U$. It follows that

$$\{0 < \varepsilon < 1 : [0, \varepsilon) \subseteq U\}$$

is not empty. We let $t_0 = \sup\{0 < \varepsilon < 1 : [0, \varepsilon) \subseteq U\}$. Notice that $t_0 \leq 1$.

1. *Claim 1*: $[0, t_0) \subseteq U$.
   Indeed, for each $n \geq 1$, let $r_n > 0$ with $t_0 - 1/n < r_n < t_0$ such that $[0, r_n) \subseteq U$. We then have

$$[0, t_0) = \bigcup_{n=1}^{\infty} [0, r_n) \subseteq U$$

2. *Claim 2*: $t_0 \notin U$.
   SFAC $t_0 \in U$, thus we obtain that $t_0 \neq 1$ because if $t_0 = 1 \in U$, then

$$\begin{aligned} U &\supseteq [0, t_0) \cup \{t_0\} \\ &= [0, 1) \cup \{1\} \\ &= [0, 1] \end{aligned}$$

   which contradicts property (c) as we simultaneously have

$$U \cap [0, 1] \cap V = \varnothing \qquad [0, 1] \cap V \neq \varnothing$$

   Therefore, there exists $\delta > 0$ such that $(t_0 - \delta, t_0 + \delta) \subseteq U$. We may assume $t_0 + \delta < 1$. Then we know that

$$[0, t_0 + \delta) \subseteq [0, t_0) \cup [t_0, t_0 + \delta) \subseteq U$$

   contradicting the definition of t_0.

Therefore we deduce that $t_0 \in V$. Since V is open, we can find $\delta_V > 0$ such that $(t_0 - \delta_V, t_0 + \delta_V) \subseteq V$. But then take some $0 < r < t_0$, $r > t_0 - \delta_V$, then $r \in [0, 1]$, and $r \in U$ by claim 1, while $r \in V$. Contradiction (see theorem 6.1). ☐

6.2 Higher-Dimensional Examples

Definition 6.2: Convex

We say that $C \subseteq \mathbb{R}^N$ is **convex** if for all $x, y \in C$, we have

$$tx + (1 - t)y \in C \qquad \forall\, t \in [0, 1]$$

In other words, C contains the line segment between any two points in C.

6.3 Convex is Connected

Theorem 6.2

Any convex set $C \subseteq \mathbb{R}^N$ is connected.

Proof. SFAC $C \in \mathbb{R}^N$ is not connected. Let $\{U, V\}$ be a disconnection. Let $x \in C \cap U$ and let $y \in C \cap V$. Define

$$\begin{aligned} U_0 &:= \{t \in \mathbb{R} : tx + (1 - t)y \in U\} \\ V_0 &:= \{t \in \mathbb{R} : tx + (1 - t)y \in V\} \end{aligned}$$

we will show that $\{U_0, V_0\}$ gives a disconnection for $[0, 1]$.
Claim: U_0 and V_0 are open.

Let $t_0 \in U_0$, so $x_0 = t_0 x + (1 - t_0)y \in U$. Since U is open, there exists $\varepsilon > 0$ such that

$$\mathcal{B}_\varepsilon(x_0) \subseteq U$$

For each $t \in \mathbb{R}$, we set $z_t := tx + (1 - t)y$. Notice that

$$\begin{aligned} \|z_t - x_0\| &= \|tx + (1 - t)y - (t_0 x + (1 - t_0)y)\| \\ &\leq \|(t - t_0)x\| + \|(t_0 - t)y\| \\ &= |t - t_0| \, \|x\| + |t - t_0| \, \|y\| \end{aligned}$$

Let $\delta > 0$, $\delta = \frac{\varepsilon}{\|x\| + \|y\|}$, then if $t \in (t_0 - \delta, t_0 + \delta)$, we get $\|z_t - x_0\| < \varepsilon$, which suggests that

$$z_t \in \mathcal{B}_\varepsilon(x_0) \subseteq U$$

This shows that $(t_0 - \delta, t_0 + \delta) \subseteq U_0$, and hence U_0 is open. Similar argument could also show that V_0 is open. Then $\{U_0, V_0\}$ is a disconnection for $[0, 1]$ because

1. $[0, 1] \subseteq U_0 \cup V_0$.
   If $t \in [0, 1]$, $z_t = tx + (1 - t)y \in C$ because we know that C is convex, thus $z_t \in U$ or $z_t \in V$. So that $z_t \in U \cup V$.
2. $[0, 1] \cap U_0 \neq \varnothing$ because $1 \in U_0$, and $[0, 1] \cap V_0 \neq \varnothing$ because $0 \in V_0$.
3. $[0, 1] \cap U_0 \cap V_0 = \varnothing$.
   Indeed, if $t \in [0, 1] \cap U_0 \cap V_0$, then $z_t \in U \cap V \cap C$ (in C because C is convex). This cannot happen because $\{U, V\}$ is a disconnection for C. Hence $[0, 1] \cap U_0 \cap V_0 = \varnothing$.

Thus $\{U, V\}$ is a disconnection for $[0, 1]$. Contradiction. □

Corollary 6.1

The following subsets of $\mathbb{R}^N$ are connected:

1. $\mathbb{R}^N$
2. open balls
3. line segments
4. subspaces

6.4 Only $\mathbb{R}^N$ and $\varnothing$ are Clopen

Corollary 6.2

The only clopen sets in $\mathbb{R}^N$ are $\mathbb{R}^N$ and $\varnothing$.

Proof. Exercise.
My Attempt:

Suppose there exists $U \subseteq \mathbb{R}^N$ with $U \neq \varnothing$ and $U \neq \mathbb{R}^N$ such that U is clopen. Thus we can find that $V := \mathbb{R}^N \setminus U$ is also clopen. Notice that thus we have

1. $\mathbb{R}^N \subseteq U \cup V$
2. $U \cap \mathbb{R}^N \neq \varnothing$ and $V \cap \mathbb{R}^N \neq \varnothing$
3. $\mathbb{R}^N \cap U \cap V = \varnothing$

which implies that $\mathbb{R}^N$ is disconnected. Contradiction. □

Lecture 10 - Monday, May 27

7 Sequence and Limits in $\mathbb{R}^N$

Definition 7.1: Sequence

A **sequence** in $\mathbb{R}^N$ is a function $f : \mathbb{N} \to \mathbb{R}^N$.
Notation: we write $x_n = f(n)$, and we write (x_n), $(x_n)_{n=1}^{\infty}$, or $(x_n)_{n\in\mathbb{N}}$ for the sequence.

Definition 7.2: Limit

We say that a sequence (x_n) in $\mathbb{R}^N$ **converges** to $a \in \mathbb{R}^N$ if for every $\varepsilon > 0$, there exists $M \in \mathbb{R}$ such that for all $n \geq M$

$$\|x_n - a\| < \varepsilon$$

or equivalently,

$$x_n \in \mathcal{B}_\varepsilon(a)$$

We call a the **limit** of (x_n) and say that (x_n) is convergent.
Notation: we write $a = \lim_{n\to\infty} x_n$, or $x_n \to a$.

Discovery 7.1

Notice that (x_n) converges to a if and only if for every open $U \subseteq \mathbb{R}^N$ with $a \in U$, there exists $M_U \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq M_U$.

Definition 7.3: Bounded

Let (x_n) be a sequence in $\mathbb{R}^N$, we say that (x_n) is **bounded** if its set of terms $\{x_n : n \in \mathbb{N}\}$ is a bounded set.

7.1 Bounded if (Cauchy iff Convergent)

Definition 7.4: Cauchy

We say (x_n) is **Cauchy** if for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \qquad \text{for all } n, m \geq M$$

Discovery 7.2

If (x_n) is a sequence in $\mathbb{R}^N$, then

$$(x_n) \text{ is convergent } \leftrightarrow (x_n) \text{ is cauchy } \rightarrow (x_n) \text{ is bounded}$$

Proposition 7.1

Let (x_n) be a sequence in $\mathbb{R}^N$, then

1. if (x_n) is convergent, then it is cauchy;
2. if (x_n) is cauchy, then it is bounded.

Proof.

1. Suppose (x_n) is convergent and let $a := \lim_{n\to\infty} x_n$. Let $\varepsilon > 0$ and let $M \in \mathbb{N}$ such that $\|x_n - a\| < \varepsilon/2$ for all $n > M$. For $m, n > M$, we have

$$\begin{aligned}\|x_n - x_m\| &\leq \|x_n - a\| + \|a - x_m\| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

thus the sequence (x_n) is cauchy.

2. Suppose (x_n) is cauchy. For $\varepsilon = 1$, let $M \in \mathbb{N}$ be such that $\|x_n - x_m\| = 1$ for all $m, n > M$, then

$$\|x_n\| = \|x_n - x_M + x_M\| \leq \|x_n - x_M\| + \|x_M\|$$

Take $R := \max\{\|x_1\|, \|x_2\|, \ldots, \|x_{M-1}\|, 1 + \|x_M\|\}$, then $\|x_n\| \leq R$ for all $n \in \mathbb{N}$, suggesting that (x_n) is bounded.

□

Proposition 7.2

A sequence (x_n) in $\mathbb{R}^N$ can have at most one limit.

Proof. Suppose (x_n) is convergent. SFAC that $a, b \in \mathbb{R}^N$, $a \neq b$ with $a = \lim_{n\to\infty} x_n = b$. Since $a \neq b$, we have $\|a - b\| \neq 0$, and we set $\varepsilon = \|a - b\|/2$. Then

$$\mathcal{B}_\varepsilon(a) \cap \mathcal{B}_\varepsilon(b) = \varnothing$$

Let $M_a \in \mathbb{N}$ be such that $x_n \in \mathcal{B}_\varepsilon(a)$ for all $n \geq M_a$, and let $M_b \in \mathbb{N}$ be such that $x_n \in \mathcal{B}_\varepsilon(b)$ for all $n \geq M_b$, then for $n \geq M := \max\{M_a, M_b\}$, we have

$$x \in \mathcal{B}_\varepsilon(a) \cap \mathcal{B}_\varepsilon(b) = \varnothing$$

which is a contradiction. □

8 Sequential Characterization of Compact Set

Proposition 8.1

Let $S \subseteq \mathbb{R}^N$ and $p \in \mathbb{R}^N$, then TFAE:

1. $p \in S'$;
2. There exists $(x_n) \in S$ with $x_n \neq x_m$ if $n \neq m$ such that $\lim_{n\to\infty} x_n = p$.

Proof. A2. □

Definition 8.1: Subsequence

A **subsequence** of a sequence (x_n) in $\mathbb{R}^N$ is a sequence of the form $(x_{n_k})_{k=1}^\infty$ with

$$n_1 < n_2 < n_3 < \cdots < n_k < \cdots$$

Example 8.1

Consider the sequence in $\mathbb{R}^3$ such that

$$x_n = \left((-1)^n, \cos\left(\frac{\pi n}{2}\right), \frac{1}{n}\right)$$

notice that it is not convergent, but it is bounded and has convergent subsequences. In particular, as for an instance, the following subsequences are convergent:

$$n_k = 2k + 1$$
$$n_k = 4k$$

Proposition 8.2

If (x_n) converges to $a \in \mathbb{R}^N$, then every subsequence also converges to a.

Proof. Let $a = \lim_{n\to\infty} x_n$ and let (x_{n_k}) be a subsequence. Let $\varepsilon > 0$ and let $M \in \mathbb{N}$ be such that

$$\|x_n - a\| < \varepsilon \qquad \text{for all } n > M$$

Let $k_0 \in \mathbb{N}$ be such that $n_{k_0} \geq M$. Then

$$k \geq k_0 \quad \Rightarrow \quad n_k \geq n_{k_0} \geq M$$

and so $\|x_{n_k} - a\| < \varepsilon$, which implies that (x_{n_k}) converges to a. □

8.1 Compact and Sequential Compact (in Metric Space $\mathbb{R}^N$)

Theorem 8.1

Let $K \subseteq \mathbb{R}^N$, TFAE:

1. K is compact;
2. Every sequence (x_n) in K has a subsequence that converges to a point in K.

Proof.

1. (1) $\Longrightarrow$ (2)
   Let (x_n) be a sequence in K, we need to consider two cases:

   (a) *Case 1*: $E := \{x_n : n \in \mathbb{N}\}$ is finite.
   Then there exists $a \in E$ such that the set $\{n \in \mathbb{N} : x_n = a\}$ is infinite. We build a subsequence (x_{n_k}) of (x_n) converging to $a \in K$ as following: We set

   $$A_1 = \{n \in \mathbb{N} : x_n = a\}$$

   then $A_1 \neq \varnothing$, and we set $n_1 = \min A_1$. Let

   $$A_2 = \{n \in \mathbb{N} : n > n_1, x_n = a\}$$

   then $A_2 \neq \varnothing$, and we set $n_2 = \min A_2$. Proceeding with the argument inductively we obtain

   $$n_1 < n_2 < \cdots < n_k < \cdots$$

   such that $x_{n_k} = a$ for all k. Thus (x_{n_k}) definitely converges to a.

   (b) *Case 2*: $E := \{x_n : n \in \mathbb{N}\}$ is infintie.
   In this case, since K is compact, then by Theorem (5.2), E has a cluster point $a \in K$. Then we build a subsequence (x_{n_k}) converging to a as following: For $\varepsilon_1 = 1$, take $x_{n_1} \in \mathcal{B}_{\varepsilon_1}(a)$; For $\varepsilon_2 = 1/2$, take $n_2 > n_1$ and $x_{n_2} \in \mathcal{B}_{\varepsilon_2}(a)$. Continue with the argument inductively, then for $\varepsilon_k = 1/n$, $n_k > n_{k-1}$ with $x_{n_k} \in \mathcal{B}_{\varepsilon_k}(a)$.

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2. (2) $\Longrightarrow$ (1)
   SFAC K is not compact, then K is either not bounded or not closed.

   (a) if K is not bounded
   So for each $n \in \mathbb{N}$, we can find $x_n \in K$ with $\|x_n\| > n$. The sequence (x_n) has no bounded subsequence, which is hence not convergent. Hence we conclude that K must be bounded.

(b) if K is not closed

By the characterization of closed set 4.1, there exists $p \in K'$ with $p \notin K$. By A02-Q4, there exists (x_n), a sequence in K, converges to p. Then every subsequence also converge to $p \notin K$ by Proposition 8.2, contradicting 2), so k must be clsoed.

□

Theorem 8.2: Bolzano-Weierstrass Theorem in $\mathbb{R}^N$

Let (x_n) be a bounded sequence in $\mathbb{R}^N$, then (x_n) has a convergent subsequence.

Proof. Suppose (x_n) is bounded, say $(x_n) \subseteq \mathcal{B}_R[0]$. Since $\mathcal{B}_R[0]$ is closed and bounded, it is compact. Hence x_n has a convergent subsequence by Theorem 8.1. □

Proof. This is an alternative proof
Using BW 8.2 in $\mathbb{R}$, since

$$x_n = (x_{n,1}, x_{n,2}, \dots, x_{n,N})$$

For the first sequence, find a convergent subsequence $(x_{n_k,1})$, and take (x_{n_k}). Using this subsequence, at the second coordinate find a convergent subsequence of (x_{n_k}), denoted as $(x_{n_{k_j},2})$, to get $(x_{n_{k_j}})$. Continuing this argument for each coordinate.

Discovery 8.1

This proof is called the "Diagonal Argument".

□

Theorem 8.3: Completeness of $\mathbb{R}^N$

Every cauchy sequence in $\mathbb{R}^N$ is convergent.

Proof. We know that by Proposition 7.1 every cauchy sequence is bounded. Let (x_n) be a cauchy sequence in $\mathbb{R}^N$. It follows by BW Theorem (8.2) in $\mathbb{R}^N$ that (x_n) has a convergent subsequence (x_{n_k}). Let $a = \lim_{k\to\infty} x_{n_k}$. We will show that (x_n) converges to a. Let $\varepsilon > 0$ and let $k_0 \in \mathbb{N}$ be such that $\|x_{n_k} - a\| < \varepsilon/2$ for all $k \geq k_0$. Let $M \in \mathbb{N}$ be such that $\|x_n - x_m\| < \varepsilon/2$ for all $n, m \geq M$. Let $n \geq M$, let k be such that $k \geq k_0$ and $n_k \geq M$ (e.g. $k \geq \max\{k_0, M\}$). Then

$$\begin{aligned}
\|x_n - a\| &= \|x_n - x_{n_k} + x_{n_k} - a\| \\
&\leq \|x_n - x_{n_k}\| + \|x_{n_k} - a\| \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

□

9 Limits of Function and Continuity

9.1 Limit

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ and $f : D \to \mathbb{R}^M$ a function, given $x_0 \in D'$, we wish to study the behaviour of f around x_0.

Definition 9.1: Limit

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ and $f : D \to \mathbb{R}^M$ a function, given $x_0 \in D'$. We say that $L \in \mathbb{R}^M$ is the **limit** of f as $x \to x_0$, written $L = \lim_{x \to x_0} f(x)$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in D$ and $0 < \|x - x_0\| < \delta$, then $\|f(x) - L\| < \varepsilon$.
If there is no $L \in \mathbb{R}^M$ such that the above happens, then we say that the limit of f at x_0 does not exist.

Theorem 9.1

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ and $f : D \to \mathbb{R}^M$ a function, given $x_0 \in D'$. TFAE:

1. $L = \lim_{x \to x_0} f(x)$
2. For every sequence (x_n) in $D \backslash \{x_0\}$ with $x_n \to x_0$, the sequence $(f(x_n))$ converges to L
3. For every neighbourhood U of L, there exists an open neighbourhood V of x_0 such that

$$(V \cap D) \backslash \{x_0\} \subseteq f^{-1}(U) := \{x \in D : f(x) \in U\}$$

Definition 9.2: Neighbourhood

U is a **neighbourhood** of x_0 if there exists $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(x_0) \subset U$.

Proof. 1. $(1 \implies 2)$
Let (x_n) be a sequence in $D \backslash \{x_0\}$ converging to x_0. Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that if $x \in D$, $0 < \|x - x_0\| < \delta$, then $\|f(x) - L\| < \varepsilon$. Let $M \in \mathbb{N}$ be such that $x_0 \in \mathcal{B}_\delta(x_0)$ for all $n \geq M$. Then $\|f(x_n) - L\| < \varepsilon$, giving that $(f(x_n))$ converges to L.

2. $(2 \implies 1)$
SFAC $L \neq \lim_{x \to x_0} f(x)$, then there exists $\varepsilon > 0$ such that for every $\delta > 0$, we can find $x_\delta \in D$ with $0 < \|x_\delta - x_0\| < \delta$ such that

$$\|f(x_\delta) - L\| > \varepsilon$$

For $\delta = 1$, find $x_1 \in \mathcal{B}_1(x_0) \backslash \{x_0\}$, $x_1 \in D$ with $\|f(x_1) - L\| > \varepsilon$. For $\delta = 1/n$, find $x_n \in D$, $x_n \in \mathcal{B}_{1/n}(x_0) \backslash \{x_0\}$ with $\|f(x_n) - L\| > \varepsilon$. The corresponding sequence $(x_n) \subseteq D \backslash \{x_0\}$ converges to x_0, but $(f(x_n))$ does not converge to L. Contradiction.

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3. $(1 \Longrightarrow 3)$
   Suppose (1) holds and let U be an open neighbourhood of L. Let $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(L) \subseteq U$. By (1), there exists $\delta > 0$ such that if $x \in D$ and $0 < \|x - x_0\| < \delta$, then $\|f(x) - L\| < \varepsilon$, which further implies that $f(x) \in \mathcal{B}_\varepsilon(L)$. Set $V := \mathcal{B}_\delta(x_0)$, then

$$(V \cap D)\backslash\{x_0\} \ \subset \ f^{-1}\left(\mathcal{B}_\varepsilon(L)\right) \ \subset \ f^{-1}(U)$$

4. $(3 \Longrightarrow 1)$
   Let $\varepsilon > 0$. Set $U := \mathcal{B}_\varepsilon(L)$. By (3) we can find an open neighbourhood V of x_0 such that

$$(V \cap D)\backslash\{x_0\} \ \subset \ f^{-1}(U)$$

   Let $\delta > 0$ be such that $\mathcal{B}_\delta(x_0) \subset V$, then if $x \in \mathcal{B}_\delta(x_0) \cap D$, $x \neq x_0$, then

$$x \in (V \cap D)\backslash\{x_0\} \quad \Rightarrow \quad x \in f^{-1}(U)$$

□

Notice: If $D \subset \mathbb{R}$, x approaches x_0 either from the left or from the right. In $\mathbb{R}^N$, $N \geq 2$, there are many different ways x can approach x_0.

Example 9.1

Consider $D = \mathbb{R}^2\backslash\{(0,0)\}$, $f : D \to \mathbb{R}$, $f(x,y) = xy/(x^2+y^2)$ and $x_0 = (0,0)$. Let (x_n) in $D\backslash\{x_0\}$, $x_n = (1/n, 1/n)$, then $x_n \to (0,0)$ and $f(x_n) \to 1/2$. Take $x_m = (1/m, 1/m^2)$, compute to find that $f(x_m) \to 0$. We conclude that by the Sequential Characterization ((2) of 9.1) that the limit of f at x_0 does not exist.

Example 9.2

Let $D = \mathbb{R}^2\backslash\{(0,0)\}$. Let $f : D \to \mathbb{R}$, $f(x,y) = x^4/(x^2+y^2)$ and $x_0 = (0,0)$. We claim that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Assume $x \neq 0$, then $f(x,y) = \frac{x^2}{1+y^2/x^2}$. We have $1 + y^2/x^2 \geq 1$, hence $\frac{1}{1+y^2/x^2} \leq 1$, giving that $f(x,y) = \frac{x^2}{1+y^2/x^2} \leq x^2$. Thus given $\varepsilon > 0$, take $\delta = \sqrt{\varepsilon}$, thus if $\|(x,y)\| < \delta$, we have $x^2 < \varepsilon$.

9.2 Continuity

Definition 9.3: Continuous

Let $D \subseteq \mathbb{R}^N$, $f : D \to \mathbb{R}^M$ be a function. We say that f is **continuous** at $x_0 \in D$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in D$ and $\|x - x_0\| < \delta$ we have $\|f(x) - f(x_0)\| < \varepsilon$. We say that f is **continuous on** D if f is continuous at every point $x_0 \in D$.

Discovery 9.1

1. Continuity only makes sense at a point $x_0 \in D$.
2. We say that a point $x_0 \in D$ is **isolated** if there exists $\delta > 0$ such that $\mathcal{B}_\delta(x_0) \cap D = \{x_0\}$ (e.g. $x_0 \in D \backslash D'$). If $x_0 \in D$ is an isolated point, then every function $f : D \to \mathbb{R}^M$ is continuous at x_0.

Theorem 9.2

Let $f : D \to \mathbb{R}^M$ be a function $x_0 \in D \cap D'$, then f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

9.3 Properties of Continuous Functions

Proposition 9.1

Let $D \subseteq \mathbb{R}^N$ and let $f, g : D \to \mathbb{R}^M$, $\phi : D \to \mathbb{R}$. Suppose f, g and ϕ are continuous at $x_0 \in D$, then

$$\begin{aligned} f + g : D &\to \mathbb{R}^M \\ x &\mapsto f(x) + g(x) \end{aligned} \qquad \begin{aligned} f \cdot g : D &\to \mathbb{R}^M \\ x &\mapsto f(x) \cdot g(x) \end{aligned} \qquad \begin{aligned} \phi f : D &\to \mathbb{R}^M \\ x &\mapsto \phi(x) \cdot f(x) \end{aligned}$$

where the second is dot product and the third is scalar multiplication, are continuous.

Proof. Exercise. (Use, for example, $f(x_n) \to f(x_0)$ if and only if $f(x_n)_j \to f(x_0)_j$ for $j = 1, \ldots, M$). □

Proposition 9.2

Let $f_1 : D_1 \to \mathbb{R}^K$, $D_1 \subseteq \mathbb{R}^N$ and $f_2 : D_2 \to \mathbb{R}^K$, $D_2 \subseteq \mathbb{R}^M$. Suppose $f_1(D_1) \subseteq D_2$. If f_1 is continuous at $x_0 \in D_1$ and f_2 is continuous at $f_1(x_0)$, then $f_2 \circ f_1 : D_1 \to \mathbb{R}^M$, $x \mapsto f_2(f_1(x))$ is continuous at x_0.

Proof. Let (x_n) be a sequence in D_1 converging to x_0. We need to show that

$$\lim_{n \to \infty} (f_2 \circ f_1)(x_n) = f_2(f_1(x_0))$$

Since f_1 is continuous at x_0, we have $(f_1(x_n))$ converges to $f_1(x_0)$. Because f_2 is continuous at $f_1(x_0)$ and $(f_1(x_n)) \to f_1(x_0)$, we get

$$\lim_{n \to \infty} f_2(f_1(x_n)) = f_2(f_1(x_0))$$

□

Proposition 9.3

Let $f : D \to \mathbb{R}^M$, $D \subseteq \mathbb{R}^N$, be a function. For each $j = 1, \ldots, M$, let $f_j : D \to \mathbb{R}$ be j^{th} component of f, so that

$$f(x) = (f_1(x), f_2(x), \ldots, f_M(x))$$

for all $x \in D$. Then f is continuous at x_0 if and only if f_j is continuous at x_0 for each j.

Proof. Exercise. $\square$

Example 9.3

For $j \in \{1, \ldots, N\}$, then the function $\pi_j : \mathbb{R}^N \to \mathbb{R}$, $(x_1, \ldots, x_N) \to x_j$ (projectino onto the j^{th} coordinate) is continuous. Then every function $f : \mathbb{R}^N \to \mathbb{R}$, $f(x_1, \ldots, x_N)x_1^{n_1} \cdots x_M^{n_N}$, $n_j \geq 0$, $j = 1, \ldots, N$ is continuous.

Example 9.4

The function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \frac{xy^2}{x^2+y^4+\pi}$ is continuous on $\mathbb{R}^2$. Indeed, $f = f_1 \cdot f_2$, for $f_1 = xy^2$ is continuous and $f_2 = \frac{1}{x^2+y^4+\pi}$ is continuous. f_2 is continuous because $f_2(x, y) = g_2 \circ g_1$ for $g_1(x, y) = x^2 + y^4 + \pi \subseteq \mathbb{R}\backslash\{0\}$ and $g_2 : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, $t \mapsto 1/t$ are continuous.

Example 9.5

The function $f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f(x, y) = \left(\cos\left(\frac{xy^2}{x^2 + y^4 + \pi}\right), \sin\left(\frac{xy^2}{x^2 + y^4 + \pi}\right), e^{x+y}\right)$$

is continuous on $\mathbb{R}^2$ since each composition f_1, f_2, f_3 of f is continuous on $\mathbb{R}$.

Global Properties of Continuity

Theorem 9.3

Let $\varnothing \neq D \subseteq \mathbb{R}^N$, $f : D \to \mathbb{R}^M$ be a function, TFAE:

1. f is continuous on D;
2. For every $U \subseteq \mathbb{R}^M$ open, there exists $V \subseteq \mathbb{R}^N$ open such that $f^{-1}(U) = V \cap D$;
3. For every $F \subseteq \mathbb{R}^M$ closed, there exists $G \subseteq \mathbb{R}^N$ closed such that $f^{-1}(F) = G \cap D$.

Proof. 1. (1) $\implies$ (2)

Suppose f is continuous on D and let $U \subseteq \mathbb{R}^M$. We claim that for each $x \in f^{-1}(U)$, there exists an open neighbourhood V_x of x such that

$$V_x \cap D \subseteq f^{-1}(U)$$

Indeed, in case that $x \in D$ is an isolated point, let $\delta_x > 0$ be such that $\mathcal{B}_{\delta_x}(x) \cap D = \{x\}$, set $V_x = \mathcal{B}_{\delta_x}(x)$. If $x \in D \cap D'$, then $\lim_{y\to x} f(y) = f(x)$. By Theorem (9.1, (1) $\to$ (3)), there exists an open neighbourhood V_x of x such that

$$(V_x \cap D)\backslash\{x\} \subseteq f^{-1}(U)$$

and hence $V_x \cap D \subseteq f^{-1}(U)$. Set $V = \bigcup_{x\in f^{-1}(U)} V_x$, then V is open in $\mathbb{R}^M$ and

$$f^{-1}(U) \subseteq \bigcup_{x\in f^{-1}(U)} V_x \cap D \subseteq f^{-1}(U)$$

giving that $f^{-1}(U) = V \cap D$.

2. (2) $\implies$ (1)

Let $x_0 \in D \cap D'$, we apply Theorem (9.1, (3) $\to$ (1)). Let U be an open neighborhood of $f(x)$. We know that there exists $V \subseteq \mathbb{R}^N$ open such that $V \cap D = f^{-1}(U)$. Then V is open neighborhood of x since $x \in f^{-1}(U)$ and $(V \cap D)\backslash\{x\} \subseteq f^{-1}(U)$. By Theorem (9.1, (3) $\to$ (1)), $\lim_{y\to x} f(y) = f(x)$, and so f is continuous at x.

3. (2) $\implies$ (3)

Suppose $F \subseteq \mathbb{R}^M$ is closed. Then F^c is open. By assumption, there exists $V \subseteq \mathbb{R}^N$ open such that

$$f^{-1}(F^c) = V \cap D$$

Now we use that $f^{-1}(F^c) = f^{-1}(F)^c \cap D$. Hence $f^{-1}(F)^c \cap D = V \cap D$. Taking complement and then the intersection with D yields $f^{-1}(F) = V^c \cap D$. Setting $G := V^c$ gives the result.

4. (3) $\implies$ (2)

Follows a similar proof as above.

□

9.3.1 Example and Application

Example 9.6

Prove that the set $F \subseteq \mathbb{R}^4$,

$$F = \left\{(x, y, z, w) : e^{x+y} \sin(zw^2) \in [0, 2], x^2 + w^2 + z^3 - y^4 \in [0, 2024]\right\}$$

is clsoed

Proof. Let $f : \mathbb{R}^4 \to \mathbb{R}^2$:

$$f(x, y, z, w) = \left(e^{x+y} \sin(zw^2), x^2 + w^2 + z^3 - y^4\right)$$

then f is continuous on $\mathbb{R}^4$, we have

$$F = f^{-1}(F') \qquad \text{where} \quad F' = [0, 2] \times [0, 2024]$$

It follows from the above Theorem (9.3, $1 \to 3$) that F is closed. □

9.4 Continuity and Compactness

Theorem 9.4

Let $\varnothing \neq K \subseteq \mathbb{R}^N$ be compact and $f : K \to \mathbb{R}^M$ be continuous on K, then $f(K)$ is compact.

Proof. Let $U = \{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $f(K)$. By Theorem (9.3) for each $\alpha \in \Lambda$, there exists $V_\alpha \subseteq \mathbb{R}^N$ open such that $V_\alpha \cap K = f^{-1}(U_\alpha)$. Set $V = \{V_\alpha\}_{\alpha\in\Lambda}$, then

$$\begin{aligned} K = f^{-1}(f(K)) = f^{-1}\left(\bigcup_{\alpha\in\Lambda} U_\alpha\right) &= \bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha) \\ &= \bigcup_{\alpha\in\Lambda} V_\alpha \cap K \subseteq \bigcup_{\alpha\in\Lambda} V_\alpha \end{aligned}$$

Hence V is an open cover for K. By compactness, V admists a finite subcover $V' = \{V_{\alpha_i} : i = 1, \ldots, l\}$. Then

$$\begin{aligned} f(K) &= f\left(\bigcup_{i=1}^{l} V_{\alpha_i} \cap K\right) \\ &= \bigcup_{i=1}^{l} f(V_{\alpha_i} \cap K) \\ &= \bigcup_{i=1}^{l} U_{\alpha_i} \cap f(K) \\ &\subseteq \bigcup_{i=1}^{l} U_{\alpha_i} \end{aligned}$$

Hence $U = \{U_{\alpha_i} : i = 1, \ldots, l\}$ is a finite subcover for $f(K)$. □

Corollary 9.1

If $\varnothing \neq K \subseteq \mathbb{R}^N$ is compact, $f : K \to \mathbb{R}^M$ be continuous, then $f(K)$ is is closed and bounded.

Proof. Theorem 5.6. □

Lecture 14 - Wednesday, Jun 5

9.4.1 Extreme Value Theorem

Theorem 9.5: Extreme Value Theorems

Suppose $\varnothing \neq K \subseteq \mathbb{R}^N$ is compact and $f : K \to R$ is continuous, then there are $x_{min}, x_{max} \in K$ such that

$$f(x_{min}) = \inf_{x \in K} f(x) \quad \text{and} \quad f(x_{max}) = \sup_{x \in K} f(x)$$

Proof. By Theorem (9.4) and Theorem (5.6), we know that $f(K)$ is closed and bounded. In particular, $\inf f(K) = \inf_{x \in K} f(x)$ and $\sup_{x \in K} f(x)$ exist. Since $f(K)$ is closed, we must have $\inf_{x \in K} \in f(K)$ and $\sup_{x \in K} \in f(K)$. □

9.5 Uniform Continuity

Definition 9.4: Uniformly continuous

Let $D \subseteq \mathbb{R}^N$ and $f : D \to \mathbb{R}^M$ be a function, we say that f is uniformly continuous if given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in D$ satisfying $\|x - y\| < \delta$, we have $\|f(x) - f(y)\| < \varepsilon$.

Example 9.7

Let $D = [-d, d] \subseteq \mathbb{R}$ be closed and bounded. Let $f : D \to \mathbb{R}$ be defined as $f(x) = x^2$. Then f is uniformly continuous on D. (In fact, D only needs to be bounded.)

Proof. $\varepsilon > 0$, we have for $x, y \in D$,

$$|f(x) - f(y)| = |x + y||x - y|$$

hence we can easily take $\delta = \varepsilon/2d$. □

Example 9.8

Let $f : (0, 1) \to \mathbb{R}$ defined as $f(x) = 1/x$, then f is not uniformly continuous on $D = (0, 1)$.

Proof. Take $\varepsilon = 1$, given $\delta > 0$, let $n \in \mathbb{N}$ be such that $\frac{1}{n} < \frac{\delta}{2}$. Set $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$. Now we have $|x - y| < \delta$, but $|f(x) - f(y)| = 1 \geq \varepsilon$. □

Example 9.9

The function $x \mapsto \sin 1/x$ $(x > 0)$ is not uniformly continuous on $(0, \infty)$ because $\lim_{x \to 0} \sin 1/x$ does not exist.

Theorem 9.6

Let $\varnothing \neq K \subseteq \mathbb{R}^N$ be compact and $f : K \to \mathbb{R}^M$ be continuous, then f is uniformly continuous on K.

Proof. SFAC that f is not. Then there exists $\varepsilon > 0$ such that for each $\delta_n = 1/n$, we can find $x_n, y_n \in K$ such that

$$\|x_n - y_n\| < \delta \qquad \|f(x_n) - f(y_n)\| \geq \varepsilon$$

Since K is compact, by Theroem (8.1), (x_n) has a subsequence (x_{n_k}) converging to a point $x \in K$. Notice that

$$\begin{aligned}\lim_{k\to\infty} y_{n_k} &= \lim_{k\to\infty} (y_{n_k} - x_{n_k} + x_{n_k}) \\ &= \underbrace{\lim_{k\to\infty} (y_{n_k} - x_{n_k})}_{\to 0} + \lim_{k\to\infty} x_{n_k} = x\end{aligned}$$

By continuity in Theorem (9.2),

$$f(x) = \lim_k f(x_{n_k}) = \lim_k f(y_{n_k})$$

then

$$\lim_k [f(x_{n_k}) - f(y_{n_k})] = 0$$

which is a contradiction. □

9.6 Continuity and Connectedness

Theorem 9.7

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be connected and $f : D \to \mathbb{R}^M$ is continuous, then $f(D)$ is connected.

Proof. SFAC $\{U, V\}$ is a disconnection for $f(D)$. Since f is continuous, by Theorem (9.3), there are open sets $\tilde{U}$ and $\tilde{V} \subseteq \mathbb{R}^N$ such that

$$f^{-1}(U) = C \cap \tilde{U} \quad \text{and} \quad f^{-1}(V) = C \cap \tilde{V}$$

Then the pair $\{\tilde{U}, \tilde{V}\}$ is a disconnection for D. Contradiction. □

9.6.1 Intermediate Value Theorem

Corollary 9.2: Intermediate Value Theorem

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be connected, $f : D \to \mathbb{R}$ be continuous. Then $f(D)$ is an interval. In particular, if $x_1, x_2 \in D$ such that $f(x_1) < c < f(x_2)$ for some $c \in \mathbb{R}$, then there exists $d \in D$ such that $f(d) = c$.

10 Differentiability on $\mathbb{R}^N$

We wish to introduce a notion of differentiability for functions $f : D \to \mathbb{R}^M$, $D \subseteq \mathbb{R}^N$ open extending the corresponding notion for real-valued functions in one variable.
Recall: If $f : (a, b) \to \mathbb{R}$ and $x_0 \in (a, b)$ then we say f is differentiable at x_0 if

$$\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

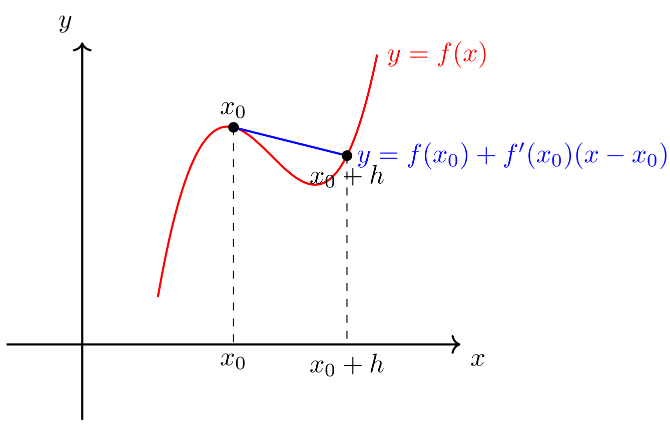
exists, and the derivative at x_0 is

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

The derivative $f'(x_0)$ gives us information such as:

- the minimum and maximum of the function,
- if the function is increasing or decreasing,
- and if $f'(x_0)$ exists then f is continuous at x_0.

The geometric intuition for a derivative is:



Here, $f'(x_0)$ is the slope of the line tangent to the graph of f at $(x_0, f(x_0))$.

Lecture 15 - Friday, Jun 7

Definition 10.1: Differentiable

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be an open set, $f : D \to \mathbb{R}^M$ be a function. We say f is **differentiable** at $x_0 \in D$ if there exists a linear transformation $T : \mathbb{R}^N \to \mathbb{R}^M$ such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - T(h)\|}{\|h\|} = 0$$

Discovery 10.1

1. The numerator we have is a norm of a vector in $\mathbb{R}^M$, and the denominator is a norm of a vector in $\mathbb{R}^N$.
2. The linear transformation $T : \mathbb{R}^N \to \mathbb{R}^M$ is a nice approximation for $f(x_0 + h) - f(x_0)$. In particular, $T(0) = f(x_0 + 0) - f(x_0) = 0$. Additionally, not only

$$\lim_{h \to 0} (f(x_0 + h) - f(x_0) - T(h)) = 0$$

but also

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - T(h)\|}{\|h\|} = 0$$

10.1 Uniqueness of Derivative

Theorem 10.1: Uniqueness of Derivative

Let $\varnothing \neq D \subseteq \mathbb{R}^N$, $f : D \to \mathbb{R}^M$ be a function. Suppose $A_1, A_2 : \mathbb{R}^N \to \mathbb{R}^M$ are linear transformations such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - A_i(h)\|}{\|h\|} = 0 \qquad \text{for } i = 1, 2$$

then $A_1 = A_2$.

Proof. For h with $x_0 + h \in D$ we have

$$\|A_1 h - A_2 h\| \leq \|A_1 h - [f(x_0 + h) - f(x_0)]\| + \|[f(x_0 + h) - f(x_0)] - A_2 h\|$$

Hence we have

$$\lim_{h \to 0} \frac{\|A_1 h - A_2 h\|}{\|h\|} = 0$$

Fix $h \in \mathbb{R}^N$, $h \neq 0$, and $t \in \mathbb{R}$, $t > 0$. By linearity, we have

$$\frac{\|A_1(th) - A_2(th)\|}{\|th\|} = \frac{\|A_1 h - A_2 h\|}{\|h\|}$$

Taking the limit of $t \to 0$, we can get that

$$\frac{\|A_1 h - A_2 h\|}{\|h\|} = \lim_{t\to 0} \frac{\|A_1(th) - A_2(th)\|}{\|th\|} = 0$$

which suggests that $A_1(h) = A_2(h)$. □

Definition 10.2: Differential

If f is differentiable at $x_0 \in D$, we call the (unique) linear transformation $T : \mathbb{R}^N \to \mathbb{R}^M$ satisfying Definition (10.1) the **differential** of f at x_0. We denote it by $(Df)(x_0)$, also $(Df)_{x_0}$ or $f'(x_0)$. Thus $Df(x_0) : \mathbb{R}^N \to \mathbb{R}^M$ is a linear transformation. We say that f is differentiable in D if f is differentiable at all $x \in D$.

Result 10.1

$$f(x_0 + h) = f(x_0) + Df(x_0) \cdot h + \text{Error}(h)$$

where

$$\lim_{h\to 0} \frac{\|\text{Error}(h)\|}{\|h\|} = 0$$

Recall from Linear Algebra. Let $\{e_1, e_2, \ldots, e_N\}$ and $\{u_1, u_2, \ldots, u_M\}$ be the standard basis for $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively. A linear transformation $T : \mathbb{R}^N \to \mathbb{R}^M$ is determined by a matrix $A \in \mathcal{M}_{M,N}(\mathbb{R})$, $A = (\alpha_{ij})$, where

$$A = \begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_N) \\ | & | & & | \end{bmatrix}$$

so that if we regard $v \in \mathbb{R}^N$ as a column vector, we have

$$T\mathbf{v} = A\mathbf{v} = A \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix}.$$

If $T : \mathbb{R}^N \to \mathbb{R}^M$ and $S : \mathbb{R}^M \to \mathbb{R}^K$, and $A \in M_{m\times n}(\mathbb{R})$ represents T and $B \in M_{m\times k}(\mathbb{R})$ represents S. Then $ST\mathbf{v} = BA\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^N$. That is, the matrix BA represents the linear transformation $ST : \mathbb{R}^N \to \mathbb{R}^K$. We have

$$\|T\| := \sup_{\|\mathbf{v}\|\le 1} \|T\mathbf{v}\| < \infty \quad \text{and} \quad \|T\mathbf{v}\| \le \|T\|\|\mathbf{v}\|$$

holds for every vector $\mathbf{v} \in \mathbb{R}^N$.

Example 10.1

Consider $N = 2$, $M = 1$ and let $D \subseteq \mathbb{R}^2$ open, $f : D \to \mathbb{R}$. Suppose that f is differentiable at $x_0 \in D$.

Then $(Df)(x_0)$ is determined by $(a, b) \in \mathcal{M}_{1,2}(\mathbb{R})$ for $a, b \in \mathbb{R}$:

$$f(x_0 + (h_1, h_2)) \approx f(x_0) + \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

$$f(x_0 + (h_1, h_2)) \approx \underbrace{f(x_0) + ah_1 + bh_2}_{\text{equation of a plane in } \mathbb{R}^3}$$

The graph of f is a surface in $\mathbb{R}^3$. Near the point $(x_0, f(x_0))$, the graph of f is approximated by the tangent plane at $(x_0, f(x_0))$.

Lecture 16 - Monday, Jun 10

Recall that if $T : \mathbb{R}^N \to \mathbb{R}^M$ is a linear transformation, then

$$\|T\| := \sup\{\|Tv\| : \|v\| \leq 1\} \ll \infty$$

Moreover,

1. $\|T\| = 0$ if and only if $T = 0$;
2. $\|\alpha T\| = |\alpha| \, \|T\|$;
3. $\|T + S\| = \|T\| + \|S\|$.

It follows that for all $h \in \mathbb{R}^N$,

$$\|T(h)\| \leq \|T\| \, \|h\|$$

because T is linear and if $\frac{h}{\|h\|}$ has norm 1, then

$$\left\| T\left(\frac{h}{\|h\|}\right) \right\| \leq \|T\| \quad \Rightarrow \quad \|T(h)\| \leq \|T\| \, \|h\|$$

Now we have the following theorem:

Theorem 10.2

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be open, $f : D \to \mathbb{R}^M$ be differentiable at $x_0 \in D$, then f is continuous at x_0.

Proof. By the definition of differentiability (10.1), we have

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - (Df)(x_0)(h)\|}{\|h\|} = 0$$

Hence we have that

$$\lim_{h \to 0} \|f(x_0 + h) - f(x_0) - (Df)(x_0)(h)\| = 0$$

Then

$$\begin{aligned} 0 &\leq \|f(x_0 + h) - f(x_0)\| \\ &\leq \|f(x_0 + h) - f(x_0) - (Df)(x_0)(h)\| + \|(Df)(x_0)(h)\| \end{aligned}$$

Taking the limit as $h \to 0$ and using that $(Df)(x_0)$ is continuous (because it is linear) yields that

$$\lim_{h \to 0} \|f(x_0 + h) - f(x_0)\| = 0$$

which suggests that f is continuous at x_0. □

Example 10.2: What is the differential of a linear tranformation $T : \mathbb{R}^N \to \mathbb{R}^M$

Suppose $N = M = 1$, $T(x) = \alpha x$ for some $\alpha \in \mathbb{R}$ for all $x \in \mathbb{R}$. Then $T'(x) = T$ is linear transformation on $\mathbb{R}$ for every $x \in \mathbb{R}$. In general for $T : \mathbb{R}^N \to \mathbb{R}^M$, we have for all $h \in \mathbb{R}^N$ and $x_0 \in \mathbb{R}^N$, we have

$$T(x_0 + h) - T(x_0) - T(h) = 0$$

In particular, $(DT)(x_0) = T$.

Example 10.3

Let $f : \mathbb{R}^N \supseteq D \to \mathbb{R}^M$ be a function and write $f = (f_1, f_2, \ldots, f_M)$, where $f_j : D \to \mathbb{R}$ for all $j = 1, 2, \ldots, M$. A linear transformation $T : \mathbb{R}^N \to \mathbb{R}^M$ is determined by the vector

$$v := T(1)$$

Then T is the differential of f at $x_0 \in D$ if and only if

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - h \cdot v\|}{\|h\|} = 0$$

It follows that f is differentiable at x_0 if and only if each component f_j is, in which case

$$(Df)(x_0) = \begin{bmatrix} f_1'(x_0) \\ f_2'(x_0) \\ \vdots \\ f_M'(x_0) \end{bmatrix}$$

determined by the derivative of its components.

10.2 Chain Rule

Theorem 10.3: Chain Rule

Suppose $\varnothing \neq D \subseteq \mathbb{R}^N$ is open. If $f : D \to \mathbb{R}^M$, $f(D) \subseteq V$, $V \subseteq \mathbb{R}^M$ is open, $g : V \to \mathbb{R}^K$. If f is differentiable at $x_0 \in D$, g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 and

$$D(g \circ f)(x_0) = (Dg)(f(x_0))(Df)(x_0)$$

Note: On the right hand side, we have the product of linear transformation $\mathbb{R}^N \to \mathbb{R}^M$ and $\mathbb{R}^M \to \mathbb{R}^K$. On the left hand side we have a function $\mathbb{R}^N \to \mathbb{R}^K$.

Proof. Let us write $y_0 = f(x_0)$,

$$A = (Df)(x_0) \quad \text{and} \quad B = (Dg)(f(x_0))$$

we wish to show that

$$\lim_{h \to 0} \frac{\|g(f(x_0+h)) - g(f(x_0)) - BA(h)\|}{\|h\|} = 0$$

We have for $h \in \mathbb{R}^N$ such that $f(x_0 + h)$ is defined,

$$g(f(x_0+h)) - g(f(x_0)) - BA(h) = g(y_0 + k) - g(y_0) - BA(h)$$

where $k = f(x_0 + h) - f(x_0)$. Since $B = (Dg)(y_0)$, given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that $g(y_0 + k')$ is defined and

$$\|g(y_0 + k') - g(y_0) - B(k')\| < \varepsilon \|k'\|$$

whenever $\|k'\| < \delta_1$. Since f is continuous at x_0, we can find $\delta_2 > 0$ such that of $h \in \mathbb{R}^N$ and $\|h\| < \delta_2$, then $f(x_0 + h)$ is defined and

$$\|k\| = \|f(x_0+h) - f(x_0)\| < \delta_1$$

Because $A = (Df)(x_0)$, we can find $\delta_3 > 0$ such that $f(x_0 + h)$ is defined and

$$\|k - A(h)\| < \varepsilon' \|h\|$$

where $\varepsilon' = \min\{\frac{\varepsilon}{\|B\|}, \varepsilon\}$. Take $\delta = \min\{\delta_2, \delta_3\}$, if $\|h\| < \delta$, then

$$\|B(k - A(h))\| \leq \|B\| \|k - A(h)\| < \varepsilon \|h\|$$

We also have

$$\|k\| < \|k - A(h)\| + \|A(h)\| < \varepsilon \|h\| + \|A\| \|h\| \tag{1}$$

and $\|k\| < \delta_1$. So we have

$$\|g(y_0 + k) - g(y_0) - BA(h)\| \leq \|g(y_0 + k) - g(y_0) - B(k)\| + \|B(k) - BA(h)\| < \varepsilon \|k\| + \varepsilon \|h\|$$

Then

$$\begin{aligned}
\frac{\|g(y_0 + k) - g(y_0) - BA(h)\|}{\|h\|} &< \frac{\varepsilon \|k\|}{\|h\|} + \varepsilon \\
&< \frac{\varepsilon(\varepsilon \|h\| + \|A\| \|h\|)}{\|h\|} + \varepsilon \\
&= \varepsilon^2 + (1 + \|A\|)\varepsilon && \text{by (1)}
\end{aligned}$$

This shows that

$$\lim_{h \to 0} \frac{\|g(y_0 + h) - g(y_0) - BA(h)\|}{\|h\|} = 0$$

□

10.3 Partial Derivative

Recall that $\{e_1, \ldots, e_N\}$ and $\{u_1, \ldots, u_M\}$ denote the standard basis of $\mathbb{R}^N$ and $\mathbb{R}^M$ respectively. For $f : D \to \mathbb{R}^M$, $\varnothing \neq D \subseteq \mathbb{R}^N$, $f = (f_1, \ldots, f_M)$ where $f_j : D \to \mathbb{R}$ is the j^{th} component of f.

Definition 10.3: Partial Derivative

For each $1 \leq i \leq N$ and $1 \leq j \leq M$, we define for $x_0 \in D$,

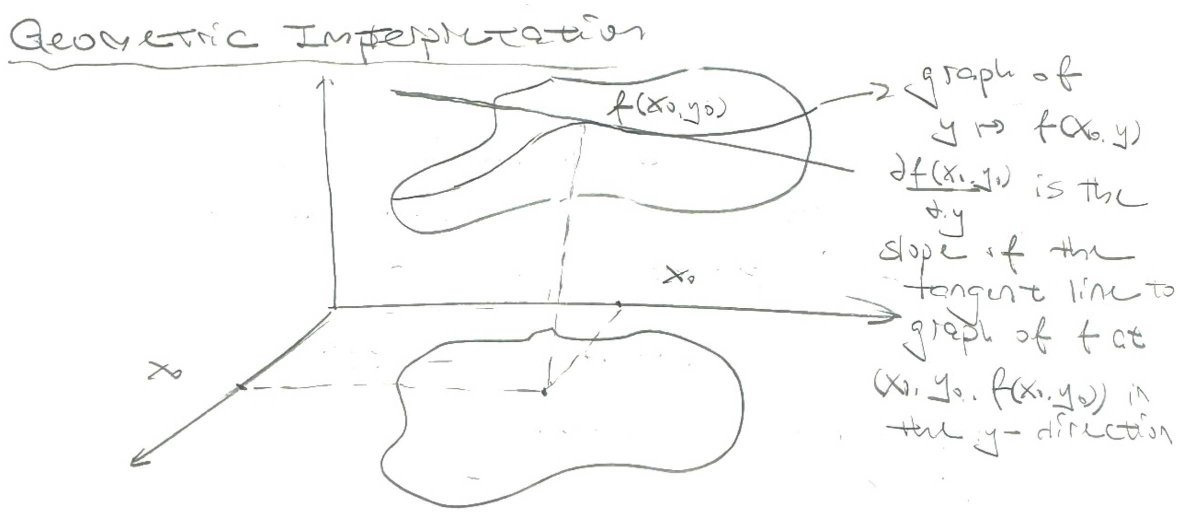
$$\frac{\partial f_j(x_0)}{\partial x_i} = \lim_{t \to 0} \frac{f_j(x_0 + te_i) - f_j(x_0)}{t}$$

provided that the limit exists. $\frac{\partial f_j(x_0)}{\partial x_i}$ is the derivative of f_j at x_0 in the x_i direction, and it is called **partial derivative** of f at x_0.
Further notation: $(D_i f_j)(x_0)$. If $M = 1$, we have $\frac{\partial f(x_0)}{\partial x_i}$, or $(D_i f)(x_0)$.

Discovery 10.2

It may happen that all partial derivative of f at x_0 exist, but f is not contimuous at x_0. But if f is differentiable at x_0, then its partial derivatives determine $(Df)(x_0)$.

10.3.1 Geometrix Interpretation



Algorithm 10.1: How do we calculate partial derivative?

We treat the variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N$ as constants.

Example 10.4

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = e^x + x\cos(xy)$, then

$$\frac{\partial f}{\partial y}(x, y) = -x^2\cos(xy) \qquad \frac{\partial f}{\partial x}(x, y) = e^x + \cos(xy) - xy\cos(xy)$$

Example 10.5: This is related to discovery (10.2)

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$. The partial derivatives of f at (x, y) exist if $(x, y) \neq (0, 0)$; If $(x, y) = (0, 0)$, we have

$$\frac{\partial f(0, 0)}{\partial x} = \lim_{t\to 0} \frac{f(t, 0) - f(0, 0)}{t} = 0 = \frac{\partial f(0, 0)}{\partial y}$$

The partial derivatives of f exist at every point, but f is not continuous at $(0, 0)$.

Recall if $T : \mathbb{R}^N \to \mathbb{R}^M$, then the matrix of T with respect to the standard basis is given by

$$\begin{bmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_N) \\ | & | & & | \end{bmatrix} = (a_{ji})_{j,i}$$

where $T(e_i) = \sum_{j=1}^{M} a_{ji}u_j$.

Theorem 10.4

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be open and $f : D \to \mathbb{R}^M$ be differentiable at $x_0 \in D$, then all the partial derivatives $\frac{\partial f_j(x_0)}{\partial x_i}$ of f at x_0 exist and

$$(Df)(x_0)(e_i) = \sum_{j=1}^{M} \frac{\partial f_j(x_0)}{\partial x_i}(u_j)$$

As a consequence, the matrix of $(Df)(x_0)$ with respect to the standard basis is given by

$$\begin{bmatrix} \frac{\partial f_1(x_0)}{\partial x_1} & \frac{\partial f_1(x_0)}{\partial x_2} & \cdots & \frac{\partial f_1(x_0)}{\partial x_N} \\ \frac{\partial f_2(x_0)}{\partial x_1} & \ddots & & \\ \vdots & & \ddots & \\ \frac{\partial f_M(x_0)}{\partial x_1} & & & \frac{\partial f_M(x_0)}{\partial x_N} \end{bmatrix} = \left(\frac{\partial f_j(x_0)}{\partial x_i}\right)_{j,i}$$

Proof. We know that

$$\lim_{t\to 0} \frac{\|f(x_0 + te_i) - f(x_0) - (Df)(x_0)(te_i)\|}{|t|} = 0$$

Using linearity of $(Df)(x_0)$, the above yields

$$\lim_{t\to 0} \frac{f(x_0 + te_i) - f(x_0)}{t} = (Df)(x_0)(e_i)$$

This implies that $\frac{\partial f_j(x_0)}{\partial x_i}$ exists for all $j = 1, \ldots, M$ and

$$(Df)(x_0)(e_i) = \left(\frac{\partial f_1(x_0)}{\partial x_i}, \ldots, \frac{\partial f_M(x_0)}{\partial x_i}\right) = \sum_{j=1}^{M} \frac{\partial f_j(x_0)}{\partial x_i}(u_j)$$

□

Definition 10.4: Jacobian Matrix

The matrix $\left[\frac{\partial f_j(x_0)}{\partial x_i}\right]_{j,i}$ is called the **Jacobian Matrix** of f at x_0 and denoted by $J_f(x_0)$.

Example 10.6

Let $\gamma : (a, b) \to D$ for $\varnothing \neq D \subseteq \mathbb{R}^N$ is open, suppose γ is differentiable in (a, b). Let $f : D \to \mathbb{R}$ be differentiable in D. Combining the chain rule (10.3) with the above theorem, we obtain that $g = f \circ \gamma$ is differentiable in (a, b) and

$$\begin{aligned} g'(t) &= (f \circ \gamma)'(t) \\ &= \left[\frac{\partial f(\gamma(t))}{\partial x_1} \cdots \frac{\partial f(\gamma(t))}{\partial x_N}\right] \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_N'(t) \end{bmatrix} = \sum_{i=1}^{N} \frac{\partial f(\gamma(t))}{\partial x_i} \gamma_i'(t) \end{aligned}$$

Definition 10.5: Gradient Notation

Let $f : D \to \mathbb{R}$ for $D \subseteq \mathbb{R}^N$ open, f differentiable at $x_0 \in D$, then $(Df)(x_0)$ is a $\mathcal{M}_{1,N}(\mathbb{R})$, $(Df)(x_0) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_N}\right)$ is called the **gradient** of f at x_0 and denoted as $\nabla f(x_0)$. Notice that if $f : D \to \mathbb{R}^M$, then

$$(Df)(x_0) = \begin{bmatrix} \nabla f_1(x_0) \\ \vdots \\ \nabla f_M(x_0) \end{bmatrix}$$

Lecture 18 - Friday, Jun 14

Definition 10.6: Directional Derivative

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be open, $f : D \to \mathbb{R}^M$ be a function. Let $x_0 \in D$ and $v \in \mathbb{R}^N$ a unit (i.e, $\|v\| = 1$). The **directional derivative** of f in the direction of v at x_0 is given by

$$(D_v f)(x_0) = \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}$$

provided that the limit exists.

Discovery 10.3

If $v = e_i$, then $(D_v f)(x_0) = \frac{\partial f}{\partial x_i}(x_0)$ is the partial derivative.

Theorem 10.5

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be open, $f : D \to \mathbb{R}$ be a function differentiable at $x_0 \in D$. Then the directional derivative of f at x_0 exists for every unit vector $v \in \mathbb{R}^N$, and

$$(D_v f)(x_0) = \nabla f(x_0) \cdot v$$

Proof. Consider the function $\gamma : \mathbb{R} \to \mathbb{R}^N$, $\gamma(t) = x_0 + tv$. Then γ is differentiable in $\mathbb{R}$ and $\gamma'(t) = v$ for all $t \in \mathbb{R}$. We have $\gamma(0) = x_0$. Since D is open, we can find $\delta > 0$ such that

$$\gamma(t) \in D \qquad \text{for all } t \in (-\delta, \delta)$$

Now

$$\begin{aligned}(D_v f)(x_0) &= \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} \\ &= \lim_{t \to 0} \frac{(f \circ \gamma)(t) - (f \circ \gamma)(0)}{t} \\ &= (f \circ \gamma)'(0)\end{aligned}$$

Example (10.6) yields

$$(f \circ \gamma)'(0) = \nabla f(\gamma(0)) \cdot \gamma'(0) = \nabla f(x_0) \cdot v$$

which is desired. □

Result 10.2

This allows for a geometric interpretation of the gradient vector. By Cauchy-Schwartz

$$\|(D_v f)(x_0)\| = \|\nabla f(x_0) \cdot v\| \leq \|\nabla f(x_0)\| \, \|v\| = \|\nabla f(x_0)\|$$

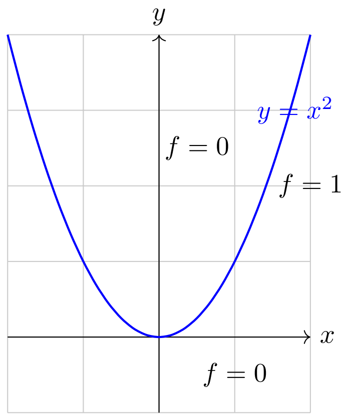
If $v = \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}$, then $\|v\| = 1$ and

$$(D_v f)(x_0) = \|\nabla f(x_0)\|$$

So the gradient of f at x_0 points in the direction to which the slope of the tangent line to the graph of f at $(x_0, f(x_0))$ is maximal.

Example 10.7: Existence of directional derivative does not imply continuity

Consider $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} 1 & 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$,



We have $(D_v f)(0, 0) = 0$ for all unit vectors $v \in \mathbb{R}^2$, but f is not continuous at $(0, 0)$.

Recall Mean Value Theorem.

Exercise: *See more at HW3.* $f : \mathbb{R}^N \to \mathbb{R}^M$ is differentiable at $x_0 \in D$ if and only if the j^{th} component of f, $f_j = \mathbb{R}^N \to \mathbb{R}$ is differentiable at x_0 for all $j = 1, \ldots, M$.

Theorem 10.6: Sufficient Condition for Differentiability

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be open, $f : D \to \mathbb{R}^M$, $x_0 \in D$. Suppose that all partial derivatives of f, $\frac{\partial f}{\partial x_i}$, exist in D and are continuous at x_0. Then f is differentiable at x_0.

Proof. We can assume $M = 1$. We know f is differentiable at x_0 if and only if

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - \nabla f(x_0) \cdot h\|}{h} = 0$$

Let $\varepsilon > 0$ be given. Since each $\frac{\partial f}{\partial x_i}$ is continuous at x_0, there exists $\delta > 0$ such that if $|z - x_0| < \delta$, then $z \in D$ and

$$\left| \frac{\partial f(z)}{\partial x_i} - \frac{\partial f(x_0)}{\partial x_i} \right| < \frac{\varepsilon}{N} \qquad i = 1, \ldots, N$$

Fix $h \in \mathbb{R}^N$ with $\|h\| < \delta$ and write $h = (h_1, \ldots, h_N)$. For each $k = 1, \ldots, N$, set

$$v_k = \sum_{i=1}^{k} h_i e_i = (h_1, \ldots, h_k, \ldots, 0_{N-k})$$

We also set $v_0 = 0$. Now $v_k = v_{k-1} + h_k e_k$ for $k = 1, \ldots, N$ and $\|v_k\| < \delta$ for all $k = 0, \ldots, N$. Now

$$f(x_0 + h) - f(x_0) - f(x_0 + v_{k-1}) + f(x_0 + v_{k-1}) = \sum_{k=1}^{N} f(x_0 + v_k) - f(x_0 + v_{k-1})$$

Fix $k = 1$, we have $x_0 + v_k, x_0 + v_{k-1} \subseteq \mathcal{B}_\delta(x_0)$. Since $\mathcal{B}_\delta(x_0)$ is convec, it follows that

$$t(x_0 + v_k) + (1-t)(x_0 + v_{k-1}) \in \mathcal{B}_\delta(x_0) \qquad \forall\, t \in [0,1]$$

For all $t \in [0,1]$,

$$x_0 + v_{k-1} + th_k e_k \in \mathcal{B}_\delta(x_0)$$

Hence the function

$$t \mapsto f(x_0 + v_{k-1} + th_k e_k)$$

is continuous on $[0,1]$ and differentiable in $(0,1)$ because $\frac{\partial f}{\partial x_k}$ exists in D. Set $g_k : [0,1] \to \mathbb{R}$, $g_k(t) = f(x_0 + v_{k-1} + th_k e_k)$, we have $g_k(1) = f(x_0 + v_k)$ and $g_k(0) = f(x_0 + v_{k-1})$. By Mean Value Theorem, there exists $c_k \in (0,1)$ such that

$$h_k \frac{\partial f}{\partial x_k}(x_0 + v_{k-1} + c_k h_k e_k) = g_k'(c_k) = f(x_0 + v_k) - f(x_0 + v_{k-1})$$

Thus

$$\begin{aligned} & f(x_0 + v_k) - f(x_0 + v_{k-1}) - \frac{\partial f(x_0)}{\partial x_k} h_k \\ =& h_k \frac{\partial f}{\partial x_k}(x_0 + v_{k-1} + c_k h_k e_k) - \frac{\partial f(x_0)}{\partial x_k} h_k \end{aligned}$$

and

$$\begin{aligned} & \left| f(x_0 + v_k) - f(x_0 + v_{k-1}) - \frac{\partial f(x_0)}{\partial x_k} h_k \right| \\ =& \left| h_k \frac{\partial f}{\partial x_k}(x_0 + v_{k-1} + c_k h_k e_k) - \frac{\partial f(x_0)}{\partial x_k} h_k \right| < h_k \cdot \frac{\varepsilon}{N} \le \|h\| \cdot \frac{\varepsilon}{N} \\ =& |h_k| \left| \frac{\partial f}{\partial x_k}(x_0 + v_{k-1} + c_k h_k e_k) - \frac{\partial f(x_0)}{\partial x_k} \right| \end{aligned}$$

now we have

$$\begin{aligned} \frac{\left| f(x_0 + h) - f(x_0) - \sum_{k=1}^N \frac{\partial f(x_0)}{\partial x_k} h_k \right|}{\|h\|} &= \frac{\left| \sum_{k=1}^N \left(f(x_0 + v_k) - f(x_0 + v_{k-1}) - \frac{\partial f(x_0)}{\partial x_k} h_k \right) \right|}{\|h\|} \\ &< \sum_{k=1}^N \frac{\|h\| \cdot \varepsilon}{\|h\| \cdot N} = \varepsilon \end{aligned}$$

□

Lecture 19 - Monday, Jun 17

Example 10.8

Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$. If $(x, y) \neq (0, 0)$, we have

$$\begin{aligned} \frac{\partial f(x, y)}{\partial x} &= 2x \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) + (x^2 + y^2) \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \left(-\frac{1}{2}\right) \frac{1}{(x^2 + y^2)^{3/2}} (2x) \\ &= 2x \sin\left(\frac{1}{\sqrt{x^2 + y^2}}\right) - \frac{x}{\sqrt{x^2 + y^2}} \cos\left(\frac{1}{\sqrt{x^2 + y^2}}\right) \end{aligned}$$

At $(0, 0)$, we have

$$\begin{aligned} \frac{\partial f(0, 0)}{\partial x} &= \lim_{(h_1,h_2)\to(0,0)} \frac{\|(h_1, h_2)\|^2 \sin\left(\frac{1}{\|(h_1,h_2)\|}\right)}{\|(h_1, h_2)\|} \\ &= \lim_{(h_1,h_2)\to(0,0)} \|(h_1, h_2)\| \sin\left(\frac{1}{\|(h_1, h_2)\|}\right) = 0 \end{aligned}$$

by squeeze theorem. This suggests that $\frac{\partial f}{\partial x}$ is continuous at every point $(x, y) \in \mathbb{R}^2 \backslash \{(0, 0)\}$, but it is not continuous at $(0, 0)$ because, for example,

$$\lim_{n\to\infty} \frac{\partial f\left(\frac{1}{2n\pi}, 0\right)}{\partial x} = -1 \neq 0 = \frac{\partial f(0, 0)}{\partial x}$$

By Theorem (10.6), f is differentiable at every point $(x, y) \neq (0, 0)$. However, f is also differentiable at $(0, 0)$:

$$\frac{\partial f(0, 0)}{\partial x} = \lim_{t\to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t\to 0} t \sin\left(\frac{1}{t}\right) = 0$$

Now we compute

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f(h_1, h_2) - f(0, 0) - 0(h_1, h_2)|}{\|(h_1, h_2)\|} = 0$$

which suggests that f is differentiable at $(0, 0)$.

10.4 Product Rule + Linearity

Proposition 10.1

Suppose $\varnothing \neq D \subseteq \mathbb{R}^N$ is open, $f, g : D \to \mathbb{R}^M$ are differentiable at $x_0 \in D$, then

$$\lambda f + g : D \to \mathbb{R}^M \qquad (\lambda f + g)(x) = \lambda f(x) + g(x)$$

is differentiable at x_0 for all $\lambda \in \mathbb{R}$, and

$$(D(\lambda f + g))(x_0) = \lambda(Df)(x_0) + (Dg)(x_0)$$

Proof. **Exercise.** □

Proposition 10.2: Product Rule

Suppose $\varnothing \neq D \subseteq \mathbb{R}^N$ is open, $f, g : D \to \mathbb{R}^M$ be functions. If f and g are differentiable at $x_0 \in D$, then

$$\underbrace{f \cdot g}_{\text{dot product}} \quad : D \to \mathbb{R} \qquad x \mapsto \underbrace{f(x) \cdot g(x)}_{\text{dot product}}$$

is differentiable at x_0, and

$$(D(f \cdot g))(x_0) = f(x_0)^T (Dg)(x_0) + g(x_0)^T (Df)(x_0)$$

In case of $M = 1$, this gives

$$\nabla(f \cdot g) = f \cdot \nabla g + g \cdot \nabla f$$

Proof. We write $v = f \cdot g = \sum_{j=1}^M f_j \cdot g_j$. If v is differentiable at x_0, then $(Dv)(x_0) = \left(\frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \dots, \frac{\partial v}{\partial x_N}\right)$. Write

$$\frac{\partial v}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\sum_{j=1}^M f_j g_j\right) = \sum_{j=1}^M \left(\frac{\partial f_j}{\partial x_i} \cdot g_j + \frac{\partial g_j}{\partial x_i} \cdot f_j\right)$$

and this is exactly the i^{th} column of $(Dv)(x_0)$, so it suffices to show that v is differentiable at x_0. We have

$$\begin{aligned}
& v(x_0 + h) - v(x_0) - (f(x_0)^T (Dg)(x_0) + g(x_0)^T (Df)(x_0)h) \\
= \;& (f \cdot g)(x_0 + h) - (f \cdot g)(x_0) - f(x_0) \cdot g(x_0 + h) + f(x_0) \cdot g(x_0 + h) \\
& - g(x_0 + h)^T (Df)(x_0)h + g(x_0 + h)^T (Df)(x_0)h \\
& - f(x_0)^T (Dg)(x_0) - g(x_0)^T (Df)(x_0)h \\
= \;& s_1 + s_2 + s_3
\end{aligned}$$

where

$$\begin{aligned}
s_1 &= (f \cdot g)(x_0 + h) - f(x_0) \cdot g(x_0 + h) - g(x_0 + h)^T (Df)(x_0) \\
s_2 &= f(x_0)g(x_0 + h) - f(x_0)g(x_0) - f(x_0)^T (Dg)(x_0)h \\
s_3 &= (g(x_0 + h) - g(x_0))^T (Df)(x_0)h
\end{aligned}$$

Then by Cauchy-Schwartz (1.1), we have

$$\begin{aligned}
\frac{|s_1|}{\|h\|} &\leq \|g(x_0 + h)\| \cdot \frac{\|f(x_0 + h) - f(x_0) - (Df)(x_0)h\|}{\|h\|}, \\
\frac{|s_2|}{\|h\|} &\leq \|f(x_0)\| \cdot \frac{\|g(x_0 + h) - g(x_0) - (Dg)(x_0)h\|}{\|h\|}, \\
\frac{|s_3|}{\|h\|} &\leq \|g(x_0 + h) - g(x_0)\| \cdot \frac{\|(Df)(x_0)h\|}{\|h\|} \\
&\leq \|g(x_0 + h) - g(x_0)\| \cdot \|(Df)(x_0)h\|
\end{aligned}$$

Since g is continuous at 0, each summation goes to 0 as $h \to 0$. □

10.5 Higher Order Partial Derivatives

Suppose $\varnothing \neq D \subseteq \mathbb{R}^N$ open and $f : D \to \mathbb{R}$,

Definition 10.7: Second Order Partial Derivative

If $i \in \{1, \ldots, N\}$ is such that $\frac{\partial f}{\partial x_i}$ exists in D, then $\frac{\partial f}{\partial x_i}$ is a function on D. If the partial derivatives of $\frac{\partial f}{\partial x_i}$ exist, we define for $j = 1, \ldots, N$,

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)$$

is called the **second order partial derivative of** f.

Definition 10.8

We say that $f \in C^0(D)$ if f is continuous on D, $f \in C^1(D)$ if $f \in C^0(D)$ and the partial derivatives of f exist in D and are continuous. If $f \in C^1(D)$, then f is continuously differentiable. In general, $f \in C^k(D)$ if $f \in C^{k-1}(D)$ and all $\frac{\partial^k f}{\partial x_{i_k} \partial x_{i_{k-1}} \cdots \partial x_{i_1}}$ are in $C^0(D)$.

Example 10.9

Suppose $f(x, y) = \frac{e^{xy}}{x}$, $(x \neq 0)$. then

$$f_x = \frac{ye^{xy}}{x} - \frac{e^{xy}}{x^2} = \left(\frac{y}{x} - \frac{1}{x^2}\right)e^{xy}$$
$$f_y = e^{xy}$$

The second order partial derivatives are

$$f_{xx} = y\left(\frac{y}{x} - \frac{1}{x^2}\right)e^{xy} + \left(\frac{-y}{x} + \frac{2}{x^3}\right)e^{xy}$$
$$f_{xy} = ye^{xy}$$
$$f_{yx} = ye^{xy}$$
$$f_{yy} = xe^{xy}$$

Discovery 10.4

Notice that $f_{xy} = f_{yx}$. In fact, partial derivatives are commutative. (See more in 10.8)

Discovery 10.5

Let $\varnothing \neq D \subseteq \mathbb{R}^N$, $N \geq 3$. Suppose $i, j \in \{1, \ldots, N\}$, $i < j$, and $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $\frac{\partial f}{\partial x_j \partial x_i}$ all exist at $x_0 = (a_1, \ldots, a_N)$. We consider $g : \mathbb{R}^2 \supseteq U \to \mathbb{R}$ defined by

$$g(x, y) = f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_{j-1}, y_j, a_{j+1}, \ldots, a_N)$$

Then we have

$$\frac{\partial g(x, y)}{\partial x} = \frac{\partial f}{\partial x_i}(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_{j-1}, y_j, a_{j+1}, \ldots, a_N)$$

This will allow us to assume $N = 2$ in the next theorem.

Theorem 10.7: Two Dimensional MVT

Let $\varnothing \neq D \subseteq \mathbb{R}^2$ be open, $f : D \to \mathbb{R}$ a function on D. Suppose $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial y \partial x}$ exist in D. Let $(a, b) \in D$, and let Q be a closed interval contained in D with opposite vertices (a, b) and $(a + h, b + k)$. Then there exists an interior point of Q, denoted as (x, y), such that

$$\Delta(f, Q) = hk \frac{\partial^2 f(x, y)}{\partial y \partial x}$$

where $\Delta(f, Q) = f(a + h, b + k) - f(a + h, b) - f(a, b + k) + f(a, b)$.

Proof. Let $v(t) := f(t, b + k) - f(t, b)$ for $t \in [a, a + h]$ (or $[a + h, a]$). Then v is differentiable in the open interval and continuous in the closed interval. By MVT, we can find x between a and $a + h$ such that

$$v'(t) = \frac{v(a + h) - v(a)}{h} = \frac{\Delta(f, Q)}{h}$$

We know that

$$\frac{\partial f(x, b + k)}{\partial x} - \frac{\partial f(x, b)}{\partial x} = v'(x)$$

Now, the function $s \mapsto \frac{\partial f(x,s)}{\partial x}$ is continuous on the interval $[b, b + k]$ (or $[b + k, b]$) and is differentiable in the open interval because $\frac{\partial^2 f}{\partial y \partial x}$ exists in D. By MVT again, we can find y between b and $b + k$ such that

$$\frac{\partial^2 f(x, y)}{\partial y \partial x} = \frac{\frac{\partial f(x,b+k)}{\partial x} - \frac{\partial f(x,b)}{\partial x}}{k}$$

Replacing the above equation with the second one, we obtain

$$\frac{\partial^2 f(x, y)}{\partial y \partial x} = \frac{\Delta(f, Q)}{hk}$$

as desired. $\square$

Lecture 21 - Friday, Jun 21

10.5.1 Partial Derivatives are Commutative

Theorem 10.8: Partial Derivatives are Commutative

Let $\varnothing \neq D \in \mathbb{R}^2$ be open, $f : F \to \mathbb{R}$. Suppose that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$ all exist in D and that $\frac{\partial^2 f}{\partial y \partial x}$ is continuous at $(a,b) \in D$. Then $\frac{\partial^2 f}{\partial x \partial y}$ exists at (a,b) and

$$\frac{\partial^2 f(a,b)}{\partial y \partial x} = \frac{\partial^2 f(a,b)}{\partial x \partial y}$$

Proof. Set $A := \frac{\partial^2 f(a,b)}{\partial y \partial x}$, we need to show that

$$\lim_{h \to 0} \left(\frac{f_y(a+h,b) - f_y(a,b)}{h} - A \right) = 0$$

Let $\varepsilon > 0$, let $\delta' > 0$ be such that if $\mathcal{B}_{\delta'}((a,b)) \subset D$ and if $(x,y) \in \mathcal{B}_{\delta'}((a,b))$, then

$$|f_{xy}(x,y) - A| < \varepsilon$$

Let $\varepsilon > 0$ such that

$$[a-\delta, a+\delta] \times [b-\delta, b+\delta] \subset \mathcal{B}_{\delta'}((a,b))$$

Take $h, k \neq 0$ with $|h|, |k| < \delta$, then the closed rectangle Q with opposite vertices (a,b) and $(a+h, b+k)$ is contained in $\mathcal{B}_{\delta'}((a,b))$. Apply Theorem (10.7), there exists $(x,y) \in D^\circ$ such that

$$\Delta(f,Q) = hk \frac{\partial^2 f}{\partial y \partial x}(x,y)$$

Then

$$\left| \frac{\Delta(f,Q)}{hk} - A \right| < \varepsilon$$

Thus

$$\left| \frac{f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)}{hk} - A \right| < \varepsilon$$

Take limit as $k \to 0$, we get

$$\left| \frac{f_y(a+h,b) - f_y(a,b)}{h} - A \right| < \varepsilon$$

since $0 \neq h \in D$, $|h| < \delta$. This shows that $f_{yx}(a,b)$ exists and

$$f_{yx}(a,b) = f_{xy}(a,b)$$

□

Corollary 10.1: Clairaut's Theorem

Let $\varnothing \neq D \in \mathbb{R}^2$ be open, $f : F \to \mathbb{R}$ in $C^2(D)$. Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \qquad \forall\, 1 \leq i, j \leq N$$

Proof. This follows Theorem (10.8) and Discovery (10.5). □

11 Vector Fields

Definition 11.1: Vector Field

A vector field is simply a function $v : \mathbb{R}^N \supset D \to \mathbb{R}^N$.

Example 11.1: Important Example

Suppose $f : D \to \mathbb{R}$ is differentiable, then

$$\nabla f : D \to \mathbb{R}^N, \quad x \in D \mapsto \left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_N}\right)$$

is a vector field called the **gradient field**.

Proposition 11.1

Suppose that $v : D \to \mathbb{R}^N$ for D open is a vector field of class 1 (in $C^1(D)$). Then a necassary condition for v to be a gradient field is that

$$\frac{\partial v_j}{\partial x_i} = \frac{\partial v_i}{\partial x_j} \qquad \forall\, 1 \leq i, j \leq N$$

Proof. Suppose $v = \nabla f$, then f must necessarily be class C^2. Then by Clairaut's Theorem (10.1),

$$\frac{\partial v_j}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial v_i}{\partial x_j}$$

□

11.1 Other Operations on a Vector Field

Definition 11.2: Divergence

Suppose $v : D \to \mathbb{R}^N$ is a differentiable vector field, then the divergence of v is

$$\begin{aligned} \operatorname{div}(v) &= \sum_{i=1}^{N} \frac{\partial v_i}{\partial x_i} = \nabla \cdot v \\ &= \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N} \right) \cdot (v_1, \ldots, v_N) \end{aligned}$$

Remark: the div corresponds to taking the trace of the Jacobian of v.

Definition 11.3: Laplace Operator

If $f : D \to \mathbb{R}$ is of class C^2, the **Laplace Operator** is

$$\Delta f = \operatorname{div}(\underbrace{\text{grad } f}_{\nabla f}) = \sum_{i=1}^{N} \frac{\partial^2 f}{\partial x_i^2}$$

Definition 11.4: Harmonic

A function $f : D \to \mathbb{R}$ is said to be **Harmonic** if $\Delta f = 0$.

The Laplace operator appears in many partial differential equation:

Example 11.2: Heat Equation and Wave Equation

Let $D \subset \mathbb{R}^N$, $f : D \times (0, \infty) \to \mathbb{R}$, $f(x, t)$ for $x \in D$ and $t \in (0, \infty)$ (think of this as "time"). The **heat** equation is

$$\frac{\partial f}{\partial t} = k \Delta f$$

The **wave** equation is

$$\frac{\partial^2 f}{\partial t^2} = k \Delta f$$

11.2 Derivative as Linear Approximation

Suppose $N = 1$. Recall that $f'(x_0)$ is the derivative of f at x_0, and

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + R_{x_0}(h),$$

for some error function $R_{x_0}(h)$, where $h = x - x_0$ and $\lim_{h\to 0} \frac{R_{x_0}(h)}{h} = 0$. If $f : D \to \mathbb{R}, D \subseteq \mathbb{R}^N, N \geq 2$ and f is differentiable at x_0, then

$$f(x) = f(x_0) + (Df)(x_0)(x - x_0) + R_{x_0}(h)$$

where $h = x - x_0$ and $\lim_{h\to 0} \frac{\|R_{x_0}(h)\|}{\|h\|} = 0$. The function $L : \mathbb{R}^N \to \mathbb{R}$,

$$L(x) = f(x_0) + (Df)(x_0)(x - x_0)$$

is the linear approximation of f at x_0. If $N = 2$, then for $(x_0, y_0) \in D$,

$$\begin{aligned} L(x) &= f(x_0, y_0) + \nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) \\ &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \end{aligned}$$

is the tangent plane to the graph of f.

Lecture 22 - Monday, Jun 24

12 Taylor's Theorem

12.1 Single Variable Taylor's Theorem

We wish to prove a version of Taylor's Theorem for functions of several variables.

Theorem 12.1: Taylor's Theorem (one variable case)

Let $n \geq 1$ and let $f : (a, b) \to \mathbb{R}$ be n-times differentiable in (a, b). Let $x_0 \in (a, b)$, then for each $x \in (a, b)$, $x \neq x_0$, there exists ξ lying between x_0 and x such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n)}(\xi)}{n!}(x - x_0)^n$$

Proof. We let $x \neq x_0$, we prove by induction on n:

1. *Base Case:*
   When $n = 1$, the statement is the MVT.

2. *Induction Step:*
   Suppose $n \geq 2$ and write

$$p(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!}(t - x_0)^k$$

for $t \in \mathbb{R}$. Set

$$M := \frac{f(x) - p(x)}{(x - x_0)^n}$$

such that $f(x) = p(x) + M(x - x_0)^n$. We need to show that $M = f^{(n)}(s)/n!$ for some s between x_0 and x. Or equivalently, $f^{(n)}(s) = n!M$. Consider $g(t) = f(t) - p(t) - M(t - x_0)^n$, then $g(x_0) = 0$. Moreover, for $k = 1, \ldots, n-1$, we have

$$g^{(k)}(x_0) = f^{(k)}(x_0) - p^{(k)}(x_0) = 0$$

because $p^{(k)}(x_0) \equiv f^{(k)}(x_0)$ for $k = 1, \ldots, n-1$. Now

$$g^{(n)}(t) = f^{(n)}(t) - n!M$$

So we need to find ξ between x_0 and x such that $g^{(n)}(\xi) = 0$. Since $g(x) = 0$ by our choice of M, by MVT, there exists x_1 between x_0 and x such that $g'(x_1) = 0$. Since $g'(x_0) = 0$ and $g'(x_1) = 0$, again, by MVT, there exists x_2 lying between x_0 and x_1 such that $g''(x_2) = 0$. Continuing with this process, after $n-1$ steps we obtain a point x_{n-1} between x_0 and x such that $g^{(n-1)}(x_{n-1}) = 0$. Since $g^{(n-1)}(x_0) = 0$, we apply MVT again and get x_n lying between x_0 and x_{n-1} such that $g^{(n)}(x_n) = 0$. Setting $\xi := x_n$, we get

$$\frac{f^{(n)}(\xi)}{n!} = M$$

□

Corollary 12.1: Second Derivative Test

Let $f \in C^2((a,b))$. Let $x_0 \in (a,b)$ be such that $f'(x_0) = 0$. Then

1. if $f''(x_0) < 0$, then x_0 is a local maximum of f;
2. if $f''(x_0) > 0$, then x_0 is a local minimum of f;

Proof. Since f'' is continuous, then there exists $\delta > 0$ such that $(x-\delta, x+\delta) \subset (a,b)$ and $f''(x) < 0$ whenever $|x - x_0| < \delta$. Now let x with $|x - x_0| < \delta$. By Taylor's Theorem, there exists ξ between x_0 and x such that

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2 \\ &= f(x_0) + \frac{f''(\xi)}{2}(x - x_0)^2 \end{aligned}$$

Since $f''(\xi) < 0$, we get $f(x) < f(x_0)$, which implies that $f(x_0)$ is a local maximum. □

12.2 Multivariable Taylor's Theorem

Definition 12.1: Notation: Multiindex

For $n \geq 0$, we let $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$ (including 0) with $\alpha_1 + \cdots + \alpha_N = N$. For $\alpha \in \mathbb{N}_0^N$, we write

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_N^{\alpha_N}$$

for $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. we define

$$|\alpha| := \alpha_1 + \cdots + \alpha_N \quad \text{and} \quad \alpha! := \alpha_1! \cdots \alpha_N!$$

For $\alpha \in \mathbb{N}_0^N$ a **multiindex**, we write

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \qquad \text{for } f \in C^{|\alpha|},\ |\alpha| \leq n$$

Example 12.1

For an example, we have

$$D^{(1,2,1)} f = \frac{\partial^4 f}{\partial x_1 \partial x_2^2 \partial x_3} \qquad \text{and} \qquad D^{(0,1,0)} = \frac{\partial f}{\partial x_2}$$

Let $(l_1, l_2, \ldots, l_n)$ be an n-tuple in $\{1, 2, \ldots, N\}^n$. For each $k = 1, \ldots, N$, we let α_k be the number of times k appears in $(l_1, \ldots, l_n)$. Then $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multiindex with $\alpha_1 + \cdots + \alpha_N = n$. If f is of class C^n, it follows from Clairaut's Theorem that

$$\frac{\partial^n f}{\partial x_{i_1} \cdots \partial x_{i_n}} = D^\alpha f$$

If $\alpha = (\alpha_1, \ldots, \alpha_N)$ be a multiindex of $\alpha_1 + \cdots + \alpha_N = n$, there are exactly $\frac{n!}{\alpha!}$ n-tuples whose associated multiindex as above is α. This follows from the multinomial theorem:

$$(x_1 + \cdots + x_N)^n = \sum_{\alpha_1 + \cdots + \alpha_N = n} \frac{n!}{\alpha!} x^\alpha$$

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Theorem 12.2: Taylor's Theorem (N-variable)

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be open, $f : D \to \mathbb{R}$, $f \in C^n(D)$ for $n \geq 1$. Let $x_0 \in D$ and let $\xi \in \mathbb{R}^N$ be such that $x_0 + t\xi \in D$ for all $t \in [0,1]$ (line segment between x_0 and $x_0 + \xi$). Then there exists $\theta \in (0,1)$ such that

$$f(x_0 + \xi) = \sum_{|\alpha| \leq n-1} \frac{D^\alpha f(x_0)}{\alpha!} \xi^\alpha + \sum_{|\alpha| = n} \frac{D^\alpha f(x_0 + \theta\xi)}{\alpha!} \xi^\alpha$$

Example 12.2

Suppose $n = 1$, then

$$f(x_0 + \xi) = f(x_0) + \sum_{i=1}^{N} \frac{\partial f(x_0 + \theta\xi)}{\partial x_i}\xi_i = f(x_0) + \nabla f(x_0 + \theta\xi) \cdot \xi$$

See more in A3.

Example 12.3

Suppose $n = 2$ and $N = 2$, then

$$\begin{aligned} f(x_0 + \xi) &= f(x_0) + \nabla f(x_0) \cdot \xi + \frac{f_{xx}(x_0 + \theta\xi)\xi_1^2}{2} + \frac{f_{yy}(x_0 + \theta\xi)\xi_2^2}{2} + f_{xy}(x_0 + \theta\xi) \cdot \xi_1\xi_2 \\ &= f(x_0) + \nabla f(x_0) \cdot \xi + \frac{1}{2}(A(x_0 + \theta\xi)\xi) \cdot \xi \end{aligned}$$

where

$$A(x_0 + \theta\xi) = \begin{bmatrix} f_{xx}(x_0 + \theta\xi) & f_{xy}(x_0 + \theta\xi) \\ f_{yx}(x_0 + \theta\xi) & f_{yy}(x_0 + \theta\xi) \end{bmatrix}$$

Before proving the Theorem, we first introduce a Lemma:

Lemma 12.1

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be open, $f : D \to \mathbb{R}$, $f \in C^n(D)$ for $n \geq 1$. Let $x_0 \in D$ and let $\xi \in \mathbb{R}^N$ be such that $x_0 + t\xi \in D$ for all $t \in [0, 1]$. Then there exists an open interval (a, b) containing $[0, 1]$ such that $g : (a, b) \to \mathbb{R}$, $g(t) = f(x_0 + t\xi)$ is in $C^n(a, b)$ and

$$g^{(n)}(t) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} D^\alpha f(x_0 + t\xi) \cdot \xi^\alpha$$

Proof. The existence of $(a, b) \supset [0, 1]$ with $x_0 + t\xi \in D$ follows because F is open and $x_0 + t\xi \in D$ for all $t \in [0, 1]$. Let us first prove by induction on n that

$$g^{(n)}(t) = \sum_{i_1,\ldots,i_n=1}^{N} \frac{\partial^n f(x_0 + t\xi)}{\partial x_{i_1} \cdots \partial x_{i_n}} \xi_{i_1} \cdots \xi_{i_n}$$

which is the sum over all n-tuples in $\{1, 2, \ldots, N\}^n$

1. For $n = 0$, there is nothing to prove.
2. For $n = 1$, since $g = f \circ \gamma$, for $\gamma : (a, b) \to \mathbb{R}^N$, $\gamma(a, b) \subset D$, and $\gamma(t) = x_0 + t\xi$, the Chain Rule (10.3) implies that g is differentiable at $t \in (a, b)$ and

$$g'(t) = \nabla f(x_0 + t\xi) \cdot \xi = \sum_{i=1}^{N} \frac{\partial f(x_0 + t\xi)}{\partial x_i}\xi_i$$

3. Now suppose $n \geq 2$ and

$$g^{(n-1)}(t) = \sum_{i_1,\ldots,i_{n-1}=1}^{N} \frac{\partial^{n-1} f(x_0 + t\xi)}{\partial x_{i_1} \cdots \partial x_{i_{n-1}}} \xi_{i_1} \cdots \xi_{i_{n-1}}$$

Then again by the Chain Rule (10.3), $g^{(n-1)}$ is differentiable at $t \in (a, b)$ and

$$\begin{aligned} g^{(n)}(t) &= \sum_{i_1,\ldots,i_{n-1}=1}^{N} \frac{d}{dt} \left(\frac{\partial^{n-1} f(x_0 + t\xi)}{\partial x_{i_1} \cdots \partial x_{i_{n-1}}} \xi_{i_1} \cdots \xi_{i_{n-1}} \right) \\ &= \sum_{i_1,\ldots,i_{n-1}=1}^{N} \frac{\partial^{n} f(x_0 + t\xi)}{\partial x_{i_1} \cdots \partial x_{i_n}} \xi_{i_1} \cdots \xi_{i_n} \end{aligned}$$

By Clairaut's Theorem (10.1), since there are exactly $\frac{n!}{\alpha!}$ n-tuples whose associated multiindex is $\alpha = (\alpha_1, \ldots, \alpha_N)$, we have

$$g^{(n)}(t) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} D^{\alpha} f(x_0 + t\xi) \cdot \xi^{\alpha}$$

□

Proof. This is the prove of N-varaible Taylor's Theorem (12.2). We need to find $\theta \in (0, 1)$ such that

$$f(x_0 + \xi) = \sum_{|\alpha| \leq n-1} \frac{D^{\alpha} f(x_0)}{\alpha!} \xi^{\alpha} + \sum_{|\alpha|=n} \frac{D^{\alpha} f(x_0 + \theta\xi)}{\alpha!} \xi^{\alpha}$$

Let (a, b) and $g : (a, b) \to \mathbb{R}$, $g(t) = f(x_0 + t\xi)$ be as in Lemma above. By the one variable Taylor's Theorem (12.1), there exists $\theta \in (0, 1)$ such that

$$g(1) = \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} (1 - 0)^k + \frac{g^{(n)}(\theta)}{n!} (1 - 0)^n = \sum_{k=0}^{n-1} \frac{g^{(k)}(0)}{k!} + \frac{g^{(n)}(\theta)}{n!}$$

Since

$$\begin{aligned} \frac{g^{(k)}(0)}{k!} &= \frac{1}{k!} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} D^{\alpha} f(x_0) \cdot \xi^{\alpha} \right) \quad (k \leq n-1) \\ \text{and} \quad \frac{g^{(n)}(0)}{n!} &= \frac{1}{n!} \left(\sum_{|\alpha|=n} \frac{n!}{\alpha!} D^{\alpha} f(x_0 + \theta\xi) \cdot \xi^{\alpha} \right) \end{aligned}$$

Substituting them in above equation we get the desired expression for $f(x_0 + \xi)$. □

12.3 Multivariate Polynomial

Definition 12.2: Multivariate Polynomial

A **multivariate polynomial** $p : \mathbb{R}^N \to \mathbb{R}$ (or N-variable) of degree n is given by

$$p(\xi) = \sum_{k=0}^{n} \left(\sum_{|\alpha|=k} C_\alpha \xi^\alpha \right)$$

where $C_\alpha \neq 0$ for some α with $|\alpha| = n$.

Discovery 12.1

Notice that

$$D^\alpha p(0) = \alpha! C_\alpha \quad \Rightarrow \quad C_\alpha = \frac{D^\alpha p(0)}{\alpha!}$$

Definition 12.3: Taylor Approximation

Suppose $f \in C^{n+1}(D)$, the n^{th} **order Taylor Approximation** of f is the polynomial

$$T_{n,x_0}(\xi) = \sum_{|\alpha| \leq n} \frac{D^\alpha f(x_0)}{\alpha!} \xi^\alpha$$

and the remainder term is $f(x_0 + \xi) - T_{n,x_0}(\xi) = \sum_{|\alpha|=n+1} \frac{D^\alpha f(x_0+\theta\xi)}{\alpha!} \xi^\alpha$.

Proposition 12.1

Let $f \in C^{n+1}(D)$, D open, $f : D \to \mathbb{R}$, let $x_0 \in D$, then

$$\lim_{\xi \to 0} \frac{|R_n(\xi)|}{\|\xi\|^n} = 0$$

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Proof. Let $r > 0$ be such that $\mathcal{B}_r[x_0] \subset D$. Since $f \in C^{n+1}(D)$ and $\mathcal{B}_r[x_0]$ is compact, we can find $M \geq 0$ such that

$$|D^\alpha f(y)| \leq M \qquad \text{for all } y \in \mathcal{B}_r[x_0]$$

and all multiindex α with $|\alpha| = n + 1$. Then if $\|\xi\| \leq r$, we have

$$\frac{|R_n(\xi)|}{\|\xi\|^n} \leq \sum_{|\alpha|=n+1} \frac{|D^\alpha f(x_0 + \theta\xi)|}{\alpha!} \frac{|\xi^\alpha|}{\|\xi\|^n} \leq \sum_{|\alpha|=n+1} \frac{M}{\alpha!} \frac{\|\xi\|^{n+1}}{\|\xi\|} = \sum_{|\alpha|=n+1} \frac{M}{\alpha!} \|\xi\|$$

□

Example 12.4

et $f(x,y) = \cos(x+2y)$ defined on $\mathbb{R}^2$, find $T_{2,(0,0)}(\xi)$
We have $f(0,0) = 1$, also

$$\begin{aligned} f_x(x,y) &= -\sin(x+2y) & f_y(x,y) &= -2\sin(x+2y) \\ f_{xx}(x,y) &= -\cos(x+2y) & f_{yy}(x,y) &= -4\cos(x+2y) \\ f_{xy}(x,y) &= -2\cos(x+2y) \end{aligned}$$

Then

$$\begin{aligned} T_{2,(0,0)}(\xi_1,\xi_2) &= f(0,0) + f_x(0,0)\xi_1 + f_y(0,0)\xi_2 + \frac{f_{xx}(0,0)}{2}\xi_1^2 + \frac{f_{yy}(0,0)}{2}\xi_2^2 + f_{xy}(0,0)\xi_1\xi_2 \\ &= 1 - \frac{\xi_1^2}{2} - \frac{4\xi_2^2}{2} - 2\xi_1\xi_2 \\ &= 1 - \frac{1}{2}\left(\xi_1^2 + 4\xi_2^2 - 4\xi_1\xi_2\right) \end{aligned}$$

12.4 The Hessian

Definition 12.4: Hessian

Let $\varnothing \neq D \subseteq \mathbb{R}^N$ be open, $f : D \to \mathbb{R}$, $f \in C^2(D)$. The **Hessian of** f **at** $x \in D$ denoted by $(\text{Hess } f)(x)$, is $N \times N$ matrix whose i,j-entry is $\frac{\partial^2 f(x)}{\partial x_i \partial x_j}$, that is

$$(\text{Hess } f)(x) = \begin{bmatrix} \frac{\partial^2 f(x)}{\partial x_1^2} & \frac{\partial^2 f(x)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x)}{\partial x_1 \partial x_N} \\ \frac{\partial^2 f(x)}{\partial x_2 \partial x_1} & \ddots & & \\ \vdots & & \ddots & \\ \frac{\partial^2 f(x)}{\partial x_N \partial x_1} & & & \frac{\partial^2 f(x)}{\partial x_N^2} \end{bmatrix}$$

Notice that $(\text{Hess } f)(x)$ is symmetric by Clairaut's Theorem (10.1).

Corollary 12.2

Let $f \in C^2(D)$, $D \subset \mathbb{R}^N$ be open. Let $x_0 \in D$ and $\xi \in \mathbb{R}^N$ be such that $x_0 + t\xi \in D$ for all $t \in [0,1]$, then there exists $\theta \in (0,1)$ such that

$$f(x_0 + t\xi) = f(x_0) + \nabla f(x_0) \cdot \xi + \frac{1}{2}\left[((\text{Hess } f)(x_0 + \theta\xi)\xi) \cdot \xi\right]$$

Proof. STP that for all $x \in D$ we have

$$\sum_{|\alpha|=2} \frac{(D^\alpha f)(x)}{\alpha!}\xi^\alpha = \frac{1}{2}\left[((\text{Hess } f)(x)\xi) \cdot \xi\right]$$

We compute,

$$\sum_{|\alpha|=2} \frac{(D^\alpha f)(x)}{\alpha!}\xi^\alpha = \sum_{i=1}^{N} \frac{f_{x_i x_j}(x_0)\xi_1^2}{2} + \sum_{i<j} f_{x_i x_j}(x)\xi_i\xi_j$$
$$= \frac{1}{2}\left(\sum_{i=1}^{N} f_{x_i x_j}(x)\xi_i^2 + \sum_{i\neq j} f_{x_i x_j}(x)\xi_i\xi_j\right)$$
$$= \frac{1}{2}\left[((\text{Hess } f)(x)\xi)\cdot\xi\right]$$

as desired. □

12.5 Critiacal Points

Definition 12.5: Stationary Point (Critiacal Point)

Let $f \in C^1(D)$, $f : D \to \mathbb{R}$,

1. we say that $x_0 \in D$ is a **stationary point** of f (or a **critical point** of f) if $\nabla f(x_0) = 0$.
2. x_0 is a local maximum if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in \mathcal{B}_\delta(x_0) \cap D$.
3. x_0 is a local minimum if there exists $\delta > 0$ such that $f(x) \geq f(x_0)$ for all $x \in \mathcal{B}_\delta(x_0) \cap D$.

Discovery 12.2

If x_0 is a local maximum (or a local minimum) of f, then x_0 is a critical point. This is becasue if $g(t) = f(x_0 + te_i)$ where $1 \leq i \leq N$, then 0 is a local maximum (or local minimum) of g and so

$$0 = g'(0) = \frac{\partial f(x_0)}{\partial x_i} \quad \Rightarrow \quad \nabla f(x_0) = 0$$

Example 12.5

Let $f(x, y) = x^2 - y^2$ defined on $\mathbb{R}^2$, then

$$\nabla f(x, y) = (2x, -2y)$$

hence $(0, 0)$ is a critial point of f, but it is neither a local maximum nor a local minimum.

Definition 12.6: Saddle Point

A critial point of f that is neither a local maximum nor a local minimum is called a **saddle point**.

In order to clarify stationary point we need more linear algebra.

Definition 12.7

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix, we say

1. A is **positive definite** if $(A\xi)\cdot\xi > 0$ for all $0 \neq \xi \in \mathbb{R}^N$;
2. A is **positive semidefinite** if $(A\xi)\cdot\xi \geq 0$ for all $\xi \in \mathbb{R}^N$;
3. A is **negative definite** if $(A\xi)\cdot\xi < 0$ for all $0 \neq \xi \in \mathbb{R}^N$;
4. A is **negative semidefinite** if $(A\xi)\cdot\xi \leq 0$ for all $\xi \in \mathbb{R}^N$;
5. A is **indefinite** if there are $x, y \in \mathbb{R}^N$ with $(Ax)\cdot x > 0$ and $(Ay)\cdot y < 0$.

Example 12.6

For an instance, $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ is positive definite, I is positive definite, and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is indefinite.

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In order to prove the Theorem (12.3), we first prove the following Lemma:

Lemma 12.2

Suppose $f \in C^2(D)$, and $x_0 \in D$ be such that $(\text{Hess } f)(x_0)$ is positive definite (or negative definite). Then there exists $\delta > 0$ such that for $x \in D$ and $x \in \mathcal{B}_\delta(x_0)$, then $(\text{Hess } f)(x)$ is positive definite (or negative definite).

Proof. We will prove the statement for $(\text{Hess } f)(x_0)$ positive definite. Write $A_x = (\text{Hess } f)(x_0)$. Define $Q : \mathbb{R}^N \to \mathbb{R}$, $Q(\xi) = (A_{x_0}\xi)\cdot\xi$. Then Q is continuous because it is the dot product of continuous functions on $\mathbb{R}^N$. For all unit vectors $\xi \in S^{N-1} = \partial\mathcal{B}_1(0)$, we have $Q(\xi) > 0$. Since S^{N-1} is compact, by the Extreme Value Theorem, there exists $r > 0$ such that $Q(\xi) \geq r$ for all $\xi \in S^{N-1}$. Since $f \in C^2(D)$, we can find $\delta > 0$ such that $\mathcal{B}_\delta(x_0) \subset D$ and

$$\sum_{i=1}^{N} |f_{x_i x_i}(x) - f_{x_i x_i}(x_0)| + \sum_{i \neq j} \left|f_{x_i x_j}(x) - f_{x_i x_j}(x_0)\right| < \frac{r}{2}$$

Then if $x \in \mathcal{B}_\delta(x_0)$, we have for $\xi \in S^{N-1}$

$$\begin{aligned} |(A_x\xi)\cdot\xi - (A_{x_0}\xi)\cdot\xi| &= \left|\sum_{i=1}^{N}(f_{x_i x_i}(x) - f_{x_i x_i}(x_0))\xi_i^2 + \sum_{i \neq j}(f_{x_i x_j}(x) - f_{x_i x_j}(x_0))\xi_i\xi_j\right| \\ &\leq \sum_{i=1}^{N} |f_{x_i x_i}(x) - f_{x_i x_i}(x_0)| + \sum_{i \neq j} \left|f_{x_i x_j}(x) - f_{x_i x_j}(x_0)\right| < \frac{r}{2} \end{aligned}$$

This implies that for $\xi \in S^{N-1}$:

$$(A_x\xi)\cdot\xi > (A_{x_0}\xi)\cdot\xi - \frac{r}{2} \geq r - \frac{r}{2} = \frac{r}{2} > 0$$

so $x \in \mathcal{B}_\delta(x_0)$, and $\xi \in \mathbb{R}^N\backslash\{0\}$ and we get

$$(A_x\xi)\cdot\xi = \|\xi\|^2\left(A_x\left(\frac{\xi}{\|\xi\|}\right)\cdot\frac{\xi}{\|\xi\|}\right) > 0$$

Hence A_x is positive definite for all $x \in \mathcal{B}_\delta(x_0)$. □

Theorem 12.3: Second Derivative Test

Let $\varnothing \neq D \subset \mathbb{R}^N$ be open and $f : D \to \mathbb{R}$, $f \in C^2(D)$. Let $x_0 \in D$ be a critical point of f, then

1. If $(\text{Hess } f)(x_0)$ is positive definite, then f has an local minimum at x_0;
2. If $(\text{Hess } f)(x_0)$ is negative definite, then f has an local maximum at x_0;
3. If $(\text{Hess } f)(x_0)$ is indefinite, then f has an saddle point at x_0;

Discovery 12.3

For an example where the above Theorem (12.3) does not apple, see A4.

Proof. 1. Suppose $(\text{Hess } f)(x_0)$ is positive definite. Let $\delta > 0$ be such that $(\text{Hess } f)(\gamma)$ is positive definite for all $\gamma \in \mathcal{B}_\delta(x_0) \subset D$. Take $x \in \mathcal{B}_\delta(x_0)$. Write $\xi := x - x_0$, so that $\|\xi\| < \delta$. By Taylor's Theorem (12.2), there exists $\theta \in (0,1)$ such that

$$\begin{aligned} f(x_0+\xi) &= f(x_0) + \nabla f(x_0)\cdot\xi + \frac{1}{2}(\text{Hess } f)(x_0+\theta\xi)\cdot\xi \\ &= f(x_0) + \frac{1}{2}\left[(\text{Hess } f)(x_0+\theta\xi)\cdot\xi\right] \end{aligned}$$

Then

$$f(x) - f(x_0) = f(x_0+\theta\xi) - f(x_0) = \frac{1}{2}(\text{Hess } f(x_0+\theta\xi)\xi)\cdot\xi > 0$$

Hence x_0 is a local minimum for f;

2. Follows as in (1);

3. Suppose $(\text{Hess } f)(x_0)$ is indefinite, we want to show that given $\varepsilon > 0$, there are $x, y \in \mathcal{B}_\varepsilon(x_0) \cap D$ such that

$$f(x) < f(x_0) < f(y)$$

Let ξ_1, ξ_2 be unit vectors in $\mathbb{R}^N$ such that

$$(\text{Hess } f)(x_0)\xi_1\cdot\xi_1 < 0 \quad \text{and} \quad (\text{Hess } f)(x_0)\xi_2\cdot\xi_2 > 0$$

Arguing as in the proof of Lemma (12.2), we can find $\delta > 0$ such that $\mathcal{B}_\delta(x_0) \subset D$ and if $x \in \mathcal{B}_\delta(x_0)$,

$$(\text{Hess } f)(x)\xi_1 \cdot \xi_1 < 0 \quad \text{and} \quad (\text{Hess } f)(x)\xi_2 \cdot \xi_2 > 0$$

Then given $\varepsilon > 0$, set $\varepsilon' = \min\{\delta, \varepsilon\}$ and let $\xi_{\varepsilon'} := \frac{\varepsilon'}{2}\xi_1$ and $\eta_{\varepsilon'} := \frac{\varepsilon'}{2}\xi_2$. So $x_0 + \xi_{\varepsilon'}, x_0 + \eta_{\varepsilon'} \in \mathcal{B}_\delta(x_0)$. By Taylor's Theorem (12.2), there are $\theta_1, \theta_2 \in (0,1)$ such that

$$f(x_0 + \xi_{\varepsilon'}) = f(x_0) + \left(\frac{\varepsilon'}{2}\right) \cdot \frac{1}{2}(\text{Hess } f)(x_0 + \xi_{\varepsilon'})\xi_1 \cdot \xi_1$$
$$f(x_0 + \eta_{\varepsilon'}) = f(x_0) + \left(\frac{\varepsilon'}{2}\right) \cdot \frac{1}{2}(\text{Hess } f)(x_0 + \eta_{\varepsilon'})\xi_2 \cdot \xi_2$$

Setting $x = x_0 + \xi_{\varepsilon'}$ and $y = x_0 + \eta_{\varepsilon'}$ we see that $x, y \in \mathcal{B}_\varepsilon(x_0)$ and by (1), $f(x) < f(x_0) < f(y)$.

□

Theorem 12.4

Let $A = (\alpha_{ij})_{i,j} \in \mathcal{M}_n(\mathbb{R})$ be symmetric. TFAE:

1. A is positive definite (or negative definite);
2. All eigenvalues of A are positive (or negative);
3. $\det \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k} \\ \alpha_{21} & \ddots & & \\ \vdots & & \ddots & \\ \alpha_{k1} & & & \alpha_{kk} \end{bmatrix} > 0 \left(\text{or } (-1)^k \det \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1k} \\ \alpha_{21} & \ddots & & \\ \vdots & & \ddots & \\ \alpha_{k1} & & & \alpha_{kk} \end{bmatrix} > 0 \right)$ for all $k = 1, \ldots, N$.

Corollary 12.3: Second Derivative Test in $\mathbb{R}^2$

Let $\varnothing \neq D \subset \mathbb{R}^2$ be open, $f : D \to \mathbb{R}$, $f \in C^2(D)$. Let $x_0 \in D$ be a critical point of f, then

1. If $f_{xx}(x_0) > 0$ and $f_{xx}(x_0)f_{yy}(x_0) - f_{xy}(x_0)^2 > 0$, then x_0 is a local minimum of f;
2. If $f_{xx}(x_0) < 0$ and $f_{xx}(x_0)f_{yy}(x_0) - f_{xy}(x_0)^2 > 0$, then x_0 is a local maximum of f;
3. If $f_{xx}(x_0)f_{xx}(x_0) - f_{xy}(x_0)^2 < 0$, then x_0 is a saddle point of f;

Proof. (1) and (2) are clear. For (3), let λ_1, λ_2 be the eigenvalues of $(\text{Hess } f)(x_0)$, then

$$f_{xx}(x_0)f_{xx}(x_0) - f_{xy}(x_0)^2 = \det((\text{Hess } f)(x_0)) = \lambda_1\lambda_2 \quad \Rightarrow \quad \lambda_1\lambda_2 < 0$$

So λ_1 and λ_2 have opposite signs. If ξ_1, ξ_2 are eigenvectors, we have $(\text{Hess } f)(x_0)\xi_1 \cdot \xi_1$ and $(\text{Hess } f)(x_0)\xi_2 \cdot \xi_2$ have opposite signs. Hence $(\text{Hess } f)(x_0)$ is indefinite. □

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Example 12.7

Let $K = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and let $f : K \to \mathbb{R}$, $f(x, y) = x^2 - xy + y^2$. Find the global maximum and minimum of f on K.

Proof. Since K is compact and f is continuous, we know from the Extreme Value Theorem that the problem has a solution. Let $D = K^\circ = \mathcal{B}_1((0, 0))$. We have $f_x = 2x - y$ and $f_y = 2y - x$. Then $(0, 0)$ is the only critial point of f in D. We have $f_{xx} = 2$, $f_{yy} = 2$, and $f_{xy} = -1$, so

$$(\text{Hess } f)(0, 0) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Then $f_{xx} > 0$, and $f_{xx}f_{yy} - f_{xy}^2 > 0$, thus $(\text{Hess } f)(0, 0)$ is positive definite. By second derivative test, f has local minimum at $(0, 0)$. Now we want to verify

$$\partial K = \{(x, y) : x^2 + y^2 = 1\} = \{(\cos\theta, \sin\theta) : 0 \leq \theta \leq 2\pi\}$$

Consider $g(0) = f(\cos\theta, \sin\theta) = \cos^2\theta - \cos\theta\sin\theta + \sin^2\theta = 1 - \cos\theta\sin\theta = 1 - \frac{\sin(2\theta)}{2}$, we have $g(0) \geq \frac{1}{2}$. Hence f attains its minimum on K at $(0, 0)$ since $f(0, 0) = 0$. We have $g'(0) = -\cos(2\theta)$. Thus the crital points of g in $(0, 2\pi)$ are $\theta_1 = \frac{\pi}{4}$, $\theta_2 = \frac{3\pi}{4}$, $\theta_3 = \frac{5\pi}{4}$, and $\theta_4 = \frac{7\pi}{4}$. Now $g''(0) = 2\sin(2\theta)$ gives that

$$g''(\theta_1) = 2 = g''(\theta_3) \qquad \text{and} \qquad g''(\theta_2) = -2 = g''(\theta_4)$$

Also $g(0) = 1 = g(2\pi)$, so θ_2 and θ_4 are local maximum of g. Compute $g(\theta_2) = \frac{3}{2} = g(\theta_4)$. It follow that f attains its maximum at $(\cos(\theta_2), \sin(\theta_2)) = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and at $(\cos(\theta_4), \sin(\theta_4)) = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ □

13 Local Properties of Continuously differentiable function

13.1 Inverse Function Theorem

Roughly, the IFT states that if $D \subset \mathbb{R}^N$, $f : D \to \mathbb{R}^N$, $f \in C^1(D, \mathbb{R}^N)$ and $(Df)(x_0)$ is invertible, then there exists an open neighborhood U of x_0 such that f is one-to-one on U, and $f^{-1} : f(U) \to \mathbb{R}^N$ is also continuously differentiable.

Definition 13.1: Contraction

Let $\varnothing \neq S \subset \mathbb{R}^N$ and $\varphi : S \to S$, we say that φ is a **contraction** if there exists $0 \leq c < 1$ such that

$$\|\varphi(x) - \varphi(y)\| \leq c \|x - y\| \qquad \forall\, x, y \in S$$

Theorem 13.1: Contradiction Mapping Principle

Let $\varnothing \neq F \subset \mathbb{R}^N$ be closed and $\varphi : F \to F$ be contraction. Then there exists a unique $x_* \in F$ such that $\varphi(x_*) = x_*$ (i.e. f has a unique fixed point $x_* \in F$).

Proof. For uniqueness, suppose x_*, y_* are fixed point of φ, then

$$\|x_* - y_*\| = \|\varphi(x_*) - \varphi(y_*)\| \leq c \|x_* - y_*\| < \|x_* - y_*\|$$

Hence we must have $x_* = y_*$. For existence of x_*, take $x_0 \in F$, define an sequence (x_n) in F recursively by setting $x_n = \varphi(x_{n-1})$ for $n \geq 1$, so we have for $n = 1$,

$$\begin{aligned} \|x_{n+1} - x_n\| = \|x_2 - x_1\| &= \|\varphi(x_1) - \varphi(x_0)\| \leq c \|x_1 - x_0\| \\ \|x_3 - x_2\| &= \|\varphi(x_2) - \varphi(x_1)\| \leq c \|x_2 - x_1\| \leq c^2 \|x_1 - x_0\| \end{aligned}$$

Continuing with this process by induction we obtain for all $n \geq 1$,

$$\|x_{n+1} - x_n\| = \|\varphi(x_n) - \varphi(x_{n-1})\| \leq c^n \|x_1 - x_0\|$$

Then if $m > n \geq 1$, we have

$$\|x_m - x_n\| = \left\| \sum_{k=n}^{m-1} (x_{k+1} - x_k) \right\| \leq \sum_{k=n}^{m-1} \|(x_{k+1} - x_k)\| \leq \sum_{k=n}^{m-1} c^k \|x_1 - x_0\|$$

Since the sum $\sum_{k=1}^{\infty} c^k \|x_1 - x_0\|$ converges because $0 \leq c < 1$, we deduce that (x_n) is a Cauchy Sequence. We let $x_* := \lim_{n\to\infty} x_n$, then $x_* \in F$ because F is closed. Since φ is continuous, we get

$$\varphi(x_*) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_{n+1} = x_*$$

proving that x_* is a fixed point of φ. $\square$

Theorem 13.2

Let $\varnothing \neq D \subset \mathbb{R}^N$ be an open convex set. Let $f : D \to \mathbb{R}^M$ be differentiable and suppose there exists $R \in \mathbb{R}$ such that $\|Df(x)\| \leq R$ for all $x \in D$. Then for all $x, y \in D$, we have

$$\|f(x) - f(y)\| \leq R \|x - y\|$$

Proof. Fix $x, y \in D$, $x \neq y$ and consider $g : D \to \mathbb{R}$, $g(z) = (f(x) - f(y)) \cdot f(z)$. Then g is differentiable and $\nabla g(z) = (f(x) - f(y))^T (Df)(z)$ by Product Rule (10.2). By A3-Q3, there exists ξ in the line segment between x, y such that

$$g(x) - g(y) = \nabla g(\xi) \cdot (x - y)$$

Thus

$$\begin{aligned} & \|f(x) - f(y)\|^2 = (f(x) - f(y))^T (Df)(\xi)(x - y) \\ \Rightarrow \quad & \|f(x) - f(y)\|^2 \leq \|f(x) - f(y)\| \, R \, \|x - y\| \end{aligned}$$

giving us $\|f(x) - f(y)\| \leq R \|x - y\|$. □

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Theorem 13.3: The Inverse Function Theorem

Let $\varnothing \neq D \subset \mathbb{R}^N$ be open and $f \in C^1(D, \mathbb{R}^N)$. Let $x_0 \in D$ be such that $(Df)(x_0)$ is invertible and set $y_0 := f(x_0)$, then

1. There exists an open set $U \subseteq D$, $V \subset \mathbb{R}^N$ with $x_0 \in U$, $y_0 \in V$, f is one-to-one on U and $V := f(U)$;
2. If $g : V \to \mathbb{R}^N$ is the inverse of f defined on V (i.e. $g(f(x)) = x$ for $x \in U$), then g is continuously differentiable and
$$(Dg)(y) = [(Df)(g(y))]^{-1}$$

Discovery 13.1

$(Df)(x_0)$ is invertible if and only if $\det(J_f(x_0)) \neq 0$.

If we write $f(x_1, \ldots, x_N) = (f_1(x_1, \ldots, x_N), \ldots, f_N(x_1, \ldots, x_N))$,

$$\begin{aligned} y_1 &= f_1(x_1, \ldots, x_N) \\ &\vdots \\ y_N &= f_N(x_1, \ldots, x_N) \end{aligned}$$

Result 13.1

Then the IFT (Inverse Function Theorem 13.3) tells us that the system given above can be solved for $x_1, \dots, x_N$ in terms of $y_1, \dots, y_N$ when we restrict to a small neighborhood of x_0 and y_0, and the solution is continuously differentiable.

Example 13.1

Let $u = \frac{x^4+y^4}{x}$ and $v = \sin x + \cos y$. Can we sovle the system above for x and y in terms of u and v? We have

$$J_f(x,y) = \begin{bmatrix} \frac{3x^4-y^4}{x^2} & \frac{4y^3}{x} \\ \cos x & -\sin y \end{bmatrix}$$

$$\Rightarrow \quad \det(J_f(x,y)) = -\sin y \left(\frac{3x^4 - y^4}{x^2} \right) - \cos x \cdot \frac{4y^3}{x}$$

If, for example, $x_0 = (x,y) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, then

$$\det(J_f(x_0)) = -\left[3\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2 \right] = -2\left(\frac{\pi}{2}\right)^2 \neq 0$$

Hence the IFT (13.3) says that near x_0 we can solve the system for x and y in terms of u and v.

Proof. This is the proof for IFT (13.3)
The formula for $(Dg)(y)$ follows from $Q5c$ in the Midterm Exam.

1. For part 1:
   Set $A = (Df)(x_0)$. Let U be an open ball such that

   $$\|(Df)(x) - A\| < \lambda \qquad \text{where } \lambda = \frac{1}{2\,\|A^{-1}\|}$$

   This exists becasue f is continuously differentiable. We can also find that $(Df)(x)$ is invertible for all $x \in U$ (See A4Q5). For $y \in \mathbb{R}^N$ fixed, define $\varphi_y : D \to \mathbb{R}^N$ by

   $$\varphi_y(x) = x + A^{-1}(y - f(x))$$

   (a) *Claim 1*: $y = f(x)$ if and only if x is a fixed point of φ_y
   Indeed, $y = f(x)$ gives $\varphi_y(x) = x$ since $A^{-1}(y - f(x)) = 0$. Conversely, if $\varphi_y(x) = x$, then $A^{-1}(y - f(x)) = 0$, which implies that $y - f(x) = 0$ because A^{-1} is one-to-one.

   (b) *Claim 2*: $\|\varphi_y(x) - \varphi_y(z)\| \leq \frac{1}{2}\|x - z\|$ for all $x, z \in U$
   Notice that $\varphi_y(x) = Ix + A^{-1}y - A^{-1}f(x)$, so by the Chain Rule (10.3), φ_y is differentiable and

   $$(D\varphi_y)(x) = I - A^{-1}(Df)(x)$$

Then

$$\|(D\varphi_y)(x)\| = \left\|A^{-1}A - A^{-1}(Df)(x)\right\| = \left\|A^{-1}(A - (Df)(x))\right\|$$
$$\leq \left\|A^{-1}\right\| \|A - (Df)(x)\|$$
$$< \left\|A^{-1}\right\| \frac{1}{2\left\|A^{-1}\right\|} = \frac{1}{2}$$

Hence by Theorem (13.2), we have $\|\varphi_y(x) - \varphi_y(z)\| \leq \frac{1}{2}\|x - z\|$ for all $x, z \in U$.

This shows that φ_y has at most one fixed point in U, so f is one-to-one in U by Claim 1. Set $V = f(U)$, we will show that V is open. Let $w \in V$ and let $z \in U$ be such that $w = f(z)$. Let $r > 0$ be such that $\mathcal{B}_z = \mathcal{B}_r(z) \subset U$, we will find $\delta > 0$ such that if $\|y - w\| < \delta$, then $\varphi_y(\mathcal{B}_z) \subset \mathcal{B}_z$. First, notice that if $x \in \mathcal{B}_z$, then by Claim 2,

$$\|\varphi_y(x) - \varphi_y(z)\| \leq \frac{1}{2}\|x - z\| = \frac{r}{2}$$

Let $\delta := \lambda r$, and let $y \in \mathbb{R}^N$, $\|y - w\| < \delta$, then

$$\|\varphi_y(z) - z\| = \left\|z + A^{-1}(y - f(z)) - z\right\| = \left\|A^{-1}(y - w)\right\| \leq \left\|A^{-1}\right\| \|y - w\| < \left\|A^{-1}\right\| \cdot \frac{r}{2\left\|A^{-1}\right\|} = \frac{r}{2}$$

Then if $\|y - w\| < \delta$, and $x \in \mathcal{B}_z$, we have

$$\|\varphi_y(x) - z\| \leq \|\varphi_y(x) - \varphi_y(z)\| + \|\varphi_y(z) - z\|$$
$$\leq \frac{r}{2} + \frac{r}{2} = r$$

giving that $\varphi_y(\mathcal{B}_z) \subset \mathcal{B}_z$. By the Contraction Mapping Principle, φ_y has a unique fixed point $x_* \in \mathcal{B}_z$, so $y = f(x_*) \in f(U) = V$ by Claim 1. This shows that $f(U)$ is open.

2. For part 2:
Let $g : V \to \mathbb{R}^N$ be the inverse of f on U. Let $y \in V$, $y + k \in V$, and let $x, x + h \in U$ be such that

$$f(x) = y, \quad f(x + h) = y + k$$

Notice that h is uniquely determined by k.

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Notice that

$$\varphi_y(x + h) - \varphi_y(x) = h + A^{-1}(y - f(x + h))$$
$$= h - A^{-1}k$$

Thus by Claim 2,

$$\left\|h - A^{-1}k\right\| \leq \frac{1}{2}\|x + h - x\| = \frac{\|h\|}{2} \quad \Rightarrow \quad \left\|A^{-1}k\right\| - \|h\| \leq \frac{\|h\|}{2}$$

giving that $\left\|A^{-1}k\right\| \geq \frac{\|h\|}{2}$. Hence

$$\|h\| \leq \left\|A^{-1}\right\| \cdot 2 \cdot \|k\| = \lambda^{-1}\|k\|$$

Let $T = [(Df)(x)]^{-1}$, then

$$\begin{aligned} g(y+k) - g(y) - Tk &= h - Tk \\ &= TT^{-1}h - Tk \\ &= T\left((Df)(x)h - (f(x+h) - f(x))\right) \end{aligned}$$

Now we have

$$\frac{\|g(y+k) - g(y) - Tk\|}{\|k\|} \leq \frac{\|T\|\,\|f(x+h) - f(x) - (Df)(x)h\|}{\lambda\,\|h\|}$$

Taking the limit of k approaches 0, then h approaches 0, and it follows that

$$\lim_{k \to 0} \frac{\|g(y+k) - g(y) - Tk\|}{\|k\|} = 0$$

proving that g is differentiable at y. Finally, we will show that $g \in C^1(V, \mathbb{R}^N)$, that is, $y \in V \mapsto J_g(y)$ is continuous. This follows because the map is the composition

$$V \longrightarrow^{g} U \longrightarrow^{J_f} \mathrm{GL}_N(\mathbb{R}) \longrightarrow^{-1} \mathrm{GL}_N(\mathbb{R})$$

All the maps are continuous (See A4Q5), hence $g \in C^1(V, \mathbb{R}^N)$.

□

Theorem 13.4: Open Mapping Theorem

Let $\varnothing \neq D \subset \mathbb{R}^N$ be open, $f \in C^1(D, \mathbb{R}^N)$. Suppose that $(Df)(x)$ is invertible for all $x \in D$, then for every $W \subset D$ open, $f(W) \subset \mathbb{R}^N$ is also open.

Proof. Exercise. □

13.2 Implicit Function Theorem

Definition 13.2: Level Curves

Let f be a function defined on $\mathbb{R}^2$, we write $z = f(x,y)$. The **level curve** of f determined by $c \in \mathbb{R}$ in the set of all points in $\mathbb{R}^2$ such that $f(x,y) = c$.

We wish to locally express the set of points $f(x,y) = 0$ as the graph of a function $y = g(x)$.

Example 13.2

$f(x,y) = x^2 - y$, so $f(x,y) = 0$ given $y = x^2$. Take $g(x) = x^2$.

Example 13.3

$f(x,y)=x^2+y^2-1$; Near $(1,0)$, we cannot express the set $f(x,y)=0$ as the graph of a function of $y=g(x)$.

Definition 13.3

We will write $(x,y)\in\mathbb{R}^{N+M}$ as

$$(x,y)=(x_1,\dots,x_N,y_1,\dots,y_M)$$

given a system of equations

$$\begin{aligned} f_1(x_1,\dots,x_N,y_1,\dots,y_M)&=0,\\ &\vdots\\ f_q(x_1,\dots,x_N,y_1,\dots,y_M)&=0 \end{aligned}$$

we want to locally express y in terms of x, so that $y_1=g_1(x_1,\dots,x_N),\dots,y_M=g_M(x_1,\dots,x_N)$.

13.2.1 The Linear Case

Suppose $f(x,y) = A\begin{bmatrix} x \\ y \end{bmatrix}$, $A \in \mathcal{M}_{M\times(N+M)}(\mathbb{R})$. In the case

$$A = \begin{bmatrix} A_x & A_y \end{bmatrix} \quad A_x \in \mathcal{M}_{M\times N}(\mathbb{R}),\ A_y \in \mathcal{M}_{M\times M}(\mathbb{R})$$

we have $f(x,y) = 0$ gives $A_x x + A_y y = 0$. From linear algebra we know that if A_y is invertible, then the equation $A_x x + A_y y = 0$ uniquely determines y in terms of x by

$$y = -A_y^{-1} A_x x$$

In general, given a linear transformation $A : \mathbb{R}^{N+M} \to \mathbb{R}^M$, we can split A into two linear transformations $A_x : \mathbb{R}^N \to \mathbb{R}^M$ and $A_y : \mathbb{R}^M \to \mathbb{R}^M$, where $A_x(x) = A(x,0)$ and $A_y(y) = A(0,y)$, so that

$$A(x,y) = A_x(x) + A_y(y)$$

If f is differentiable, $A = J_f(x_0)$, write $A_x = \frac{\partial f}{\partial x}$, $A_y = \frac{\partial f}{\partial y}$.

Theorem 13.5: Implicit Function Theorem

Let $\varnothing \neq D \subset \mathbb{R}^{N+M}$ be open and $f \in C^1(D, \mathbb{R}^M)$. Let $(x_0, y_0) \in \mathbb{R}^{N+M}$ be such that $f(x_0, y_0) = 0$ and let $A = (Df)(x_0, y_0)$. Suppose that A_y is invertible, i.e.

$$\det \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_M} \\ \vdots & & \\ \frac{\partial f_M}{\partial y_1} & \cdots & \frac{\partial f_M}{\partial y_M} \end{bmatrix} \neq 0 \qquad \text{at } (x_0, y_0).$$

Then there exists an open neighbourhood $U \subset D$ of (x_0, y_0) and $W \subset \mathbb{R}^N$, open neighbourhood of x_0, such that

1. For every $x \in W$, there exists a unique y_x such that $(x, y_x) \in U$ such that $f(x, y_x) = 0$.
2. If we define $g : W \to \mathbb{R}^M$, $g(x) = y$, where y is as in part (a), then g is continuously differentiable ($g \in C^1(W, \mathbb{R}^M)$), $(x,y) \in U$ and $f(x,y) = 0$, $\forall x \in W$, and

$$(Dg)(x_0) = -A_y^{-1} A_x$$

Discovery 13.2

The function g is implicitly defined by the equation $f(x,y) = 0$.

Proof. Define $F := D \to \mathbb{R}^{N+M}$ by $F(x,y) = (x, f(x,y))$. Then F is continuously differentiable because f is. Our claim is that $(DF)(x_0, y_0)$ is invertible. Indeed, we have

$$J_F(x_0,y_0) = \begin{bmatrix} I_N & 0_{N\times M} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

Then because A_y is invertible,

$$\det J_F(x_0,y_0) = \det I_N \cdot \det \frac{\partial f}{\partial y} \neq 0$$

Then the Inverse Function Theorem (13.3) gives us an open neighborhood $U \subset D$ of (x_0, y_0) such that $V := F(U)$ is open, F is one-to-one on U, and $G : V \to U \subset \mathbb{R}^{N+M}$ is also continuously differentiable.
We define $W \subset \mathbb{R}^N$ by $W := \{w \in \mathbb{R}^N : (x, 0) \in V\}$, then $x_0 \in W$ because (x_0, y_0) is in U and $F(x_0, y_0) = (x_0, 0_M)$. Also, W is open because V is open. If $x \in W$, then because $V = F(U)$, there exists $(x', y') \in U$ such that $F(x', y') = (x', f(x', y')) = (x, 0)$, which shows that $x' = x$ and $f(x, y') = 0$.
Now we wish to show uniqueness. Suppose $y_1, y_2 \in \mathbb{R}^M$ are such that $(x, y_1), (x, y_2) \in U$ and $f(x, y_1) = f(x, y_2) = 0$. It follows that $F(x, y_1) = (x, 0_M) = F(x, y_2)$. Because F is one-to-one on U, thus we must have $y_1 = y_2$, proving part (a).
For part (b), let $g : W \to \mathbb{R}^M$, $g(x) = y$. Consider $G(x, 0) = (x, g(x))$, since $G \in C^1(V, \mathbb{R}^{N+M})$ (is continuous differentiable), we must have that $g \in C^1(W, \mathbb{R}^M)$. Then we compute $(Dg)(x_0)$. Consider $\phi : W \to \mathbb{R}^{N+M}$, $\phi(x) = (x, g(x))$, then $\phi \in C^1(W, \mathbb{R}^N + M)$, $\phi(x_0) = (x_0, y_0)$. Also, for all $x \in W$ and $h \in \mathbb{R}^N$

$$(D\phi)(x)h = (h, Dg(x)h)$$

In terms of the Jacobian Matrix of ϕ at x,

$$J_\phi(x) = \begin{bmatrix} I_N \\ J_g(x) \end{bmatrix}$$

Now $f(\phi(x)) = 0$ for all $x \in W$. Applying the Chain Rule (10.3) we get

$$(Df)(\phi(x))(D\phi)(x) = 0 \qquad \forall x \in W$$

Thus for $x = x_0$ and $h \in \mathbb{R}^N$, $(Df)(x_0, y_0)(D\phi)(x_0) = 0$, add

$$\begin{aligned} (Df)(x_0,y_0)(D\phi)(x_0)h &= 0 \\ (Df)(x_0,y_0)(h, (Dg)(x_0)h) &= 0 \\ \Rightarrow \quad A_x h + A_y (Dg)(x_0)h &= 0 \end{aligned}$$

Since $A = (Df)(x_0, y_0)$, this yields

$$(Dg)(x_0)h = -A_y^{-1} A_x h$$

because A_y is invertible. Hence $(Dg)(x_0) = -A_y^{-1} A_x$ as needed. □

Discovery 13.3

Above we only needed A_y invertible to obtain $(Dg)(x_0) = -A_y^{-1}A_x$. Since the set of invertible linear tranformations is open, we can assume that $\frac{\partial f}{\partial y}$ is invertible for all $(x, y) \in U$ and hence

$$(Dg)(x) = -\left(\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial f}{\partial x} \qquad \forall x \in W$$

Example 13.4

Consider the system of equations,

$$\begin{cases} 2e^{y_1} + y_2x_1 - 4x_2 + 3 & = 0 \\ y_2 \cos y_1 - 6y_1 + 2x_1 - x_3 & = 0 \end{cases}$$

where there are five variables and two equations:

$$N + M = 5, \qquad M = 2$$

It is easy to check that $(3, 2, 7, 0, 1)$ is a solution. Can we solve the solution near $(3, 2, 7, 0, 1)$ by $(x, g(x))$ where $g : W \to \mathbb{R}^2$, $W \subset \mathbb{R}^3$.
Let $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(x_1, x_2, x_3, y_1, y_2) = (f_1(x, y), f_2(x, y))$ where

$$f_1(x, y) = 2e^{y_1} + y_2x_1 - 4x_2 + 3$$
$$f_2(x, y) = y_2 \cos y_1 - 6y_1 + 2x_1 - x_3$$

We have $f \in C^1(\mathbb{R}^5, \mathbb{R}^2)$ and

$$J_f(x, y) = \begin{bmatrix} y_2 & -4 & 0 & 2e^{y_1} & x_1 \\ 2 & 0 & -1 & -y_2 \sin y_1 - 6 & \cos y_1 \end{bmatrix}$$

At $(3, 2, 7, 0, 1)$

$$J_f(3, 2, 7, 0, 1) = \begin{bmatrix} 1 & -4 & 0 & 2 & 3 \\ 2 & 0 & -1 & -6 & 1 \end{bmatrix}$$

Hence

$$A_x = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}, \qquad A_y = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}$$

Now $\det A_y = 2 + 18 = 20 \neq 0$, so A_y is invertible. Thus by the Implicit Function Theorem (13.5), there exists an open neighborhood $W \subset \mathbb{R}^3$, of $(3, 2, 7)$, and $g : W \to \mathbb{R}^2$, continuously differentiable with $g(3, 2, 7) = (0, 1)$. Also

$$f(x, g(x)) = 0 \qquad \forall x \in W$$

We have $(Dg)(3,2,7) = -A_y^{-1}A_x$, where $A_y^{-1} = \frac{1}{20}\begin{bmatrix}1 & -3\\ 6 & 2\end{bmatrix}$, thus

$$(Dg)(3,2,7) = \begin{bmatrix}\frac{1}{4} & \frac{1}{5} & \frac{3}{20}\\ -\frac{1}{2} & \frac{6}{5} & \frac{1}{10}\end{bmatrix}$$

This does not give the partial derivative of g at $(3,2,7)$.

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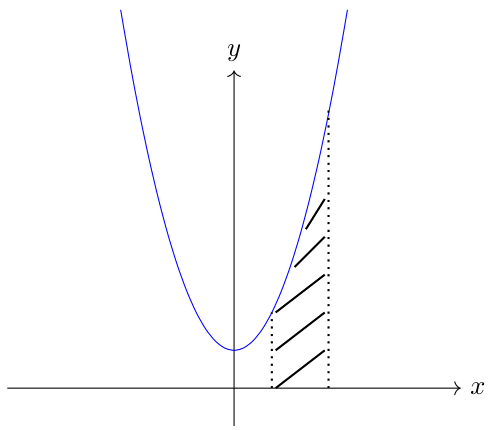
14 Integration on $\mathbb{R}^N$

Suppose $f : [a,b] \to \mathbb{R}$, $f \geq 0$, f is Riemann Integrable. Then

$$\int_a^b f\ dx$$

represents the area under the graph of f:



$\displaystyle\int f\,\mathrm{d}x$ is defined as the limit of Riemann Sums, so that

$$\int f\,\mathrm{d}x \approx \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$$

Suppose $f : [a,b]\times[c,d] \to \mathbb{R}$, $f(x) = e$, $e \geq 0$, then we expect the $\displaystyle\int f$ to be the "volume" under the graph of f, so that

$$\int f = e\cdot(b-a)\cdot(d-c)$$

We wish to define th Riemann integral of $f : A \to \mathbb{R}$, $f \geq 0$ via a limit process.

We start by considering function defined on rectangles

$$I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$$

Definition 14.1: Volumn (Content)

We define the **volume** of I (also called the **content** of I) by

$$\mu(I) = Vol(I) = \prod_{i=1}^{N} (b_i - a_i)$$

Definition 14.2: Partition

For each $j = 1, \ldots, N$, let $a = t_{j,0} < t_{j,1} < \cdots < t_{j,n_j} = b_j$ be a partition of the closed interval $[a_j, b_j]$, and define

$$P_j = \{t_{j,l} : l = 0, \ldots, n_j\}$$

Then the Cartesian Product $P = P_1 \times \cdots \times P_N$ is called a **partition of** I. A partition P of I gives the subdivision of I into $n_1 \times \cdots \times n_N$ subrectangles, which are called the subrectangles corresponding to P. So for each j and $1 \leq k_j \leq N$, we have a subrectangle

$$I = [t_{1,k_1-1}, t_{1,k_1}] \times [t_{2,k_2-1}, t_{2,k_2}] \times \cdots \times [t_{N,k_N-1}, t_{N,k_N}]$$

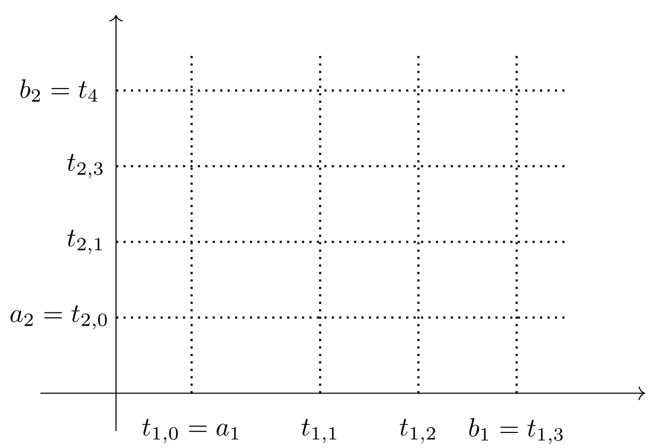


Figure 1: Subdivision generated by a partition

14.1 Riemann Sum

Definition 14.3: Riemann Sum

Let $I = [a_1, b_1] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$ be a rectangle and $f : I \to \mathbb{R}^M$ be a function. Let P be a partition of I. For each rectangle I_α in the subdivision of I corresponding to P choose $x_\alpha \in I_\alpha$, then the sum

$$S(f, P) := \sum_{\alpha \in P} f(x_\alpha)\mu(I_\alpha)$$

is called the **Riemann Sum** of f corresponding to P.

Discovery 14.1

Notice that the sum $S(f, P)$ depends on the partition P and also on the choice of points $x_\alpha \in I_\alpha$.

Definition 14.4: Refinement

Let $P = P_1 \times \cdots \times P_N$ be a partition of I, we say that a partition Q is a **refinement** of P if $P_j \subset Q_j$ for all $j = 1, \ldots, N$.

Discovery 14.2

Suppose P is a partition of I, then

$$I = \bigcup_{\alpha \in P} I_\alpha \quad \text{and} \quad \mu(I) = \sum_{\alpha \in P} \mu(I_\alpha)$$

Proof. Prove this by induction on N. The result holds because the rectangles I_α's may overlap at most along their boundaries, so Q is a refinement of P, then for each $\alpha \in P$,

$$I_\alpha = \bigcup_{\substack{\beta \in Q \\ J_\beta \subset I_\alpha}} \quad \text{and so} \quad \mu(I_\alpha) = \sum_{\substack{\beta \in Q \\ J_\beta \subset I_\alpha}} \mu(J_\beta)$$

□

Discovery 14.3

Suppose P and Q are partitions of I, then there is always a **common** refinement R of P and Q. For example,

$$R = R_1 \times \cdots \times R_N$$

where $R_j := P_j \cup Q_j$ for $j = 1, \ldots, N$.

14.2 Riemann Integrable

Definition 14.5: Riemann integrable

Let $I \subset \mathbb{R}^N$ be a rectangle and $f : I \to \mathbb{R}^M$ be a function. Suppose that there exists $y \in \mathbb{R}^M$ such that for every $\varepsilon > 0$, there exists a partition P_ε of I such that for each refinement P of P_ε and all Riemann sums $S(f, P)$ corresponding to P, we have

$$\|S(f,P) - y\| < \varepsilon$$

Then we say that f is **Riemann integrable** and y is the Riemann integral of f.

Notation:

$$y = \int_I f \qquad \int f \, \mathrm{d}\mu \qquad \int_I f(x_1, \ldots, x_N) \, \mathrm{d}\mu(x_1, \ldots, x_N)$$

Proposition 14.1

Suppose $f : I \to \mathbb{R}^M$ is Riemann integrable, then $\int_I f$ is unique.

Proof. **Exercise.** (The proof uses the uniqueness of limit). □

14.2.1 Cauchy Criterion for Riemann Integrable

Theorem 14.1: Cauchy Criterion for Riemann integrable

Let $I \subset \mathbb{R}^N$ be a rectangle and $f : I \to \mathbb{R}^M$, TFAE:

1. f is Riemann integrable;

2. For every $\varepsilon > 0$, there exists a partition P_ε such that for all refinement P and Q of P_ε and all Riemann sum $S(f, P)$ and $S(f, Q)$ corresponding to P and Q respectively, we have

$$\|S(f,P) - S(f,Q)\| < \varepsilon$$

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Proof. 1. ($\Longrightarrow$)

Given $\varepsilon > 0$, let P_ε be a partition of I such that

$$\left\| S(f,P) - \int_I f \right\| < \frac{\varepsilon}{2}$$

for all refinements P of P_ε and Riemann sums $S(f, P)$. Thus if P and Q are refinements of P_ε and $S(f, P)$ and $S(f, Q)$ are Riemann sums corresponding to P and Q respectively, we have

$$\|S(f,P) - S(f,Q)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

2. $(\Longleftarrow)$
Suppose 2. holds. Then for every $\varepsilon = \frac{1}{2^n}$ there exists a partition P_n of I such that

$$\|S(f,P) - S(f,Q)\| < \frac{1}{2^n}$$

for all refinements P and Q of P_n, and all Riemann sums $S(f,P)$ and $S(f,Q)$. By taking common refinements if necessary, we may assume that P_{n+1} is a refinement of P_n and in particular

$$\|S(f,P_{n+1}) - S(f,P_n)\| < \frac{1}{2^n}$$

for all Riemann sums corresponding to P_n and P_{n+1} respectively. For each n let y_n be a Riemann sum corresponding to the subdivision of I given by P_n. Thus $\|y_{n+1} - t_n\| < \frac{1}{2^n}$ for all n. It follows that (y_n) is a Cauchy sequence. Set $y := \lim_{n\to\infty} y_n$. We will show that $y = \int_I f$. Let $\varepsilon > 0$ be given. Choose k such that $\|y - y_n\| < \frac{\varepsilon}{2}$ for all $n \geq k$. Let $n \geq k$ such that $\frac{1}{2^n} < \frac{\varepsilon}{2}$. Set $P_\varepsilon := P_n$. Let P be a refinement of P_n and $\mathcal{S}(f,P)$ be a Riemann sum. By (10), $\|\mathcal{S}(f,P) - y_n\| < \frac{1}{2^n} < \frac{\varepsilon}{2}$. Thus

$$\|\mathcal{S}(f,P) - y\| < \|\mathcal{S}(f,P) - y_n\| + \|y_n - y\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

giving that $y = \int_I f$, and f is Riemann integrable. □

Discovery 14.4

Let $I \subset \mathbb{R}^N$ be a rectangle and $f : I \to \mathbb{R}^M$ be a function. Then f is Riemann integrable if and only if each component $f_j : I \to \mathbb{R}$, $j = 1, \dots, M$ of f is Riemann integrable (see A5).

Corollary 14.1

Let $I \subset \mathbb{R}^N$ be a rectangle and $f : I \to \mathbb{R}^M$ be a function. TFAE:

1. f is Riemann integrable;
2. For every $\varepsilon > 0$, there exists a partition P_ε of I such that

$$\|S_1(f,P_\varepsilon) - S_2(f,P_\varepsilon)\| < \varepsilon$$

   for all Riemann sums $S_1(f,P_\varepsilon)$ and $S_2(f,P_\varepsilon)$ corresponding to P_ε.

Proof. 1. $\Longrightarrow$ 2. is by *Theorem 5.7.*
2. $\Longrightarrow$ 1. Suppose 2. holds. By the preceding remark, we may assume $M = 1$. Let $\varepsilon > 0$ be given and let P_ε be a partition of I as in 2. Let P and Q be refinements of P_ε and let

$$\mathcal{S}(f,P) = \sum_{\beta \in P} f(x_\beta)\mu(J_\beta) \quad \text{and} \quad \mathcal{S}(f,Q) = \sum_{\gamma \in Q} f(x_\gamma)\mu(K_\gamma)$$

be Riemann sums associated to P and Q respectively. Then for each $\alpha \in P_\varepsilon$ we have

$$I_\alpha = \bigcup_{\beta \in P, J_\beta \subseteq I_\alpha} J_\beta = \bigcup_{\gamma \in Q, K_\gamma \subseteq I_\alpha} K_\gamma$$

and

$$\mu(I_\alpha) = \sum_{\beta \in P, J_\beta \subseteq I_\alpha} \mu(J_\beta) = \sum_{\gamma \in Q, K_\gamma \subseteq I_\alpha} \mu(K_\gamma)$$

by Discovery (14.2). For each $\alpha \in P_\varepsilon$ let

$$B_\alpha = \{f(x_\beta) \mid \beta \in P, J_\beta \subseteq I_\alpha\} \cup \{f(x_\gamma) \mid \gamma \in Q, K_\gamma \subseteq I_\alpha\}$$

Then B_α is finite and we let $z_\alpha, w_\alpha \in I_\alpha$ such that

$$f(z_\alpha) = \max B_\alpha, \quad f(w_\alpha) = \min B_\alpha.$$

Then

$$\begin{aligned} f(w_\alpha) \leq f(x_\beta) \leq f(z_\alpha), \quad \forall \beta \in P, J_\beta \subseteq I_\alpha \\ f(w_\alpha) \leq f(x_\gamma) \leq f(z_\alpha), \quad \forall \gamma \in P, K_\gamma \subseteq I_\alpha. \end{aligned}$$

We have

$$\begin{aligned} \mathcal{S}(f, P) - \mathcal{S}(f, Q) &= \sum_{\beta \in P} f(x_\beta)\mu(J_\beta) - \sum_{\gamma \in Q} f(x_\gamma)\mu(K_\gamma) \\ &= \sum_{\alpha \in P_\varepsilon} \left(\sum_{\beta \in P, J_\beta \subseteq I_\alpha} f(x_\beta)\mu(J_\beta) - \sum_{\gamma \in Q, K_\gamma \subseteq I_\alpha} f(x_\gamma)\mu(K_\gamma) \right) \\ &\leq \sum_{\alpha \in P_\varepsilon} \left(f(z_\alpha) \sum_{\beta \in P, J_\beta \subseteq I_\alpha} \mu(J_\beta) - f(w_\alpha) \sum_{\gamma \in Q, K_\gamma \subseteq I_\alpha} \mu(K_\gamma) \right) \\ &= \sum_{\alpha \in P_\varepsilon} f(z_\alpha)\mu(I_\alpha) - \sum_{\alpha \in P_\varepsilon} f(w_\alpha)\mu(I_\alpha) \\ &= S_1(f, P_\varepsilon) - S_2(f, P_\varepsilon) < \varepsilon. \end{aligned}$$

Similarly,

$$S(f, P) - S(f, Q) \leq S_1(f, P_\varepsilon) - S_2(f, P_\varepsilon) > -\varepsilon \implies \|S(f, P) - S(f, Q)\| < \varepsilon$$

by Theorem (14.1) (2. $\implies$ 1.), f is Riemann integrable. □

Theorem 14.2

Let $I \subset \mathbb{R}^N$ be a rectangle and $f : I \to \mathbb{R}^M$ be continuous. Then f is Riemann integrable.

Proof. Since I is compact and f is continuous, then f is uniformly continuous on I. Given $\varepsilon > 0$, let $\delta > 0$ be such that

$$\|f(x) - f(y)\| < \frac{\varepsilon}{\mu(I)}$$

for all $x, y \in I$, $\|x - y\| < \delta$. Choose a partition P_ε of I such that $x, y \in I_\alpha$, $\|x - y\| < \delta$ for all $\alpha \in P_\varepsilon$. Let

$$S_1(f, P_\varepsilon) = \sum_{\alpha \in P_\varepsilon} f(x_\alpha)\mu(I_\alpha), \quad S_2(f, P_\varepsilon) = \sum_{\alpha \in P_\varepsilon} f(y_\alpha)\mu(I_\alpha)$$

be Riemann sums corresponding to P_ε. Then

$$\begin{aligned}
\|S_1(f, P_\varepsilon) - S_2(f, P_\varepsilon)\| &= \left\| \sum_{\alpha \in P_\varepsilon} (f(x_\alpha) - f(y_\alpha))\mu(I_\alpha) \right\| \\
&\le \sum_{\alpha \in P_\varepsilon} \|f(x_\alpha) - f(y_\alpha)\| \mu(I_\alpha) \\
&< \sum_{\alpha \in P_\varepsilon} \frac{\varepsilon}{\mu(I)} \mu(I_\alpha) \\
&= \varepsilon
\end{aligned}$$

since $x_\alpha, y_\alpha \in I_\alpha \implies \|x_\alpha - y_\alpha\| < \delta$. By Corollary (14.1), f is Riemann integrable. □

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14.3 Content Zero

Definition 14.6: Content Zero

We say that a set $A \subset \mathbb{R}^N$ has **content zero**, write $\mu(A) = 0$, if for every $\varepsilon > 0$, the rectangle $I_1, \ldots, I_n$ (may overlap, finitely many) with

$$A \subset \bigcup_{j=1}^{n} I_j \qquad \text{and} \qquad \sum_{j=1}^{n} \mu(I_j) < \varepsilon$$

Note: if $A \subset B$ and B has a content zero, then A has content zero.

Example 14.1: Examples of content zero

1. Finite set;
2. If $A_1, \ldots, A_m$ have content zero, then their union has content zero;
3. If $I \subset \mathbb{R}^N$ is a rectangle, then ∂I has content zero. This is because ∂I is a finite union of sets of the form $[a, b] \times \cdots \times [a_{i-1}, b_{i-1}] \times \{c_i\} \times [a_{i+1}, b_{i+1}] \times [a_n, b_n]$, where $c_i \in [a_i, b_i]$.

Proposition 14.2

Suppose $K \subset \mathbb{R}^N$ is compact and $f : K \to \mathbb{R}$ is continuous, then $\text{graph}(f) = \{(x, f(x)) : x \in K\} \subset \mathbb{R}^{N+1}$ has content zero.

Proof. See A5. □

Example 14.2: Examples of non-content zero

1. $\mathbb{Z}$;
2. $\mathbb{Q}$;
3. $\mathbb{Q} \cap [0, 1]$.

14.4 Measure Zero

Definition 14.7: Measure Zero

Let $A \subset \mathbb{R}^N$, we say that A has **measure zero** if for every $\varepsilon > 0$, there are countably many (possibly infinite) rectangles $I_1, I_2, \ldots$ in $\mathbb{R}^N$ such that

$$A \subset \bigcup_{j=1}^{\infty} I_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} \mu(I_j) < \varepsilon$$

Discovery 14.5

1. $A \subset B$ and B has measure zero implies that A has measure zero;
2. A has content zero implies A has measure zero; (How does this work? Choose all the subsequent rectangles to be $\varnothing$, iykyk :3).

Proposition 14.3

Suppose $A_1, A_2, \ldots, A_n, \ldots$ are subsets of $\mathbb{R}^N$ with measure zero, then $A = \bigcup_{i=1}^{\infty} A_i$ has measure zero.

Proof. Let $\varepsilon > 0$. For each $i = 1, \ldots$, let $I_{i,1}, I_{i,2}, \ldots$ be a coutable collection of ractangles such that

$$A_i \subset \bigcup_{j=1}^{\infty} I_{i,j} \qquad \text{and} \qquad \sum_{j=1}^{\infty} \mu[I_{i,j}] < \frac{\varepsilon}{2^i}$$

Then

$$A \subset \bigcup_{i=1}^{\infty} \left(\bigcup_{j=1}^{\infty} I_{i,j} \right) \qquad \text{and} \qquad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mu[I_{i,j}] < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$$

Since $\mathbb{N} \times \mathbb{N}$ is countable, we get A has measure zero. □

Example 14.3

Countable set have measure zero (e.g. $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{Q} \cap [0, 1]$), while $[0, 1] \backslash \mathbb{Q}$ does not have measure zero.

Theorem 14.3

Suppose $K \subset \mathbb{R}^N$ is compact and has measure zero, then K has content zero.

Proof. Let $\varepsilon > 0$ and let $I_1, I_2, \ldots$ be rectangles with

$$K \subset \bigcup_{j=1}^{\infty} I_j \qquad \text{and} \qquad \sum_{j=1}^{\infty} \mu[I_j] < \frac{\varepsilon}{2}$$

For each j choose I_j' a rectangle with $I_j'^{\circ} \supset I_j$ and

$$\mu(I_j') < \mu(I_j) + \frac{\varepsilon}{2^j}$$

By compactness, there are rectangles $I_{j1}', \ldots, I_{jn}'$ such that

$$K \subset \bigcup_{i=1}^{n} I_{ji}'^{\circ} \subset \bigcup_{i=1}^{n} I_{ji}'$$

$$\sum_{i=1}^{n} \mu(I_{ji}') \leq \sum_{j=1}^{\infty} \mu(I_j)' \leq \sum_{j=1}^{\infty} \left(\mu(I_j) + \frac{\varepsilon}{2^j}\right) < \varepsilon$$

□

Definition 14.8: "Has Content"

1. Let $\varnothing \neq D \subset \mathbb{R}^N$ be bounded and let $I \subset \mathbb{R}^N$ be rectangles containing D. We say that a function $f : D \to \mathbb{R}^M$ is Riemann integrable on D if the $\overline{f} : I \to \mathbb{R}^M$ given by $\overline{f}(x) = \{f(x) : x \in D \text{ or } 0 : x \text{ otherwise}\}$ is Riemann integrable, in which case we define the integral of f on D by

$$\int_D f = \int_I \overline{f}$$

2. Let $\varnothing \neq D \subset \mathbb{R}^N$ be bounded, we say that D **Has Content** if the **Characteristic Function** on D is integrable, where

$$\mathcal{X}_D : \mathbb{R}^N \to \mathbb{R}^M, \;\; \mathcal{X}_D(x) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$

We define the content of D (the volume) by

$$\mu(D) = \int_D \mathcal{X}_D = \int_D 1$$

Discovery 14.6

If $D = I$ is rectangle, then it coincides with the volume of I.

14.5 Lebesgue Theorem

Theorem 14.4: Lebesgue Theorem

Let $I \subset \mathbb{R}^N$ be a rectangle and let $f : I \to \mathbb{R}^M$ be bounded, then f is Riemann integrable if and only if the set $B_f = \{x \in I : f \text{ is not countinuous at } x\}$ has measure zero.

Proof. Notice that we may assume $M = 1$ because

$$B_f = \bigcup_{j=1}^{M} B_{f_j}$$

where $B_{f_j} = \{x \in I : f_j \text{ is not countinuous at } x\}$, f_j is component of f.

1. $(\Longleftarrow)$
   We define for $x \in I$ the **ocsillation** of f at x by

   $$\mathfrak{o}(f,x) = \lim_{\delta \to 0} [M(x,f,\delta) - m(x,f,\delta)]$$

   where $M(x,f,\delta) = \sup\{f(y) : y \in \mathcal{B}_\delta(x)\}$ and $m(x,f,\delta) = \inf\{f(y) : y \in \mathcal{B}_\delta(x)\}$. The limit above exists because the function

   $$\delta \mapsto M(x,f,\delta) - m(x,f,\delta)$$

   is decreasing. Notice also $\mathfrak{o}(f,x) \geq 0$.

   (a) *Claim 1*: f is continuous at x if and only if $\mathfrak{o}(f,x) = 0$;

   (b) *Claim 2*: For every $\varepsilon > 0$ the set $B_\varepsilon = \{x \in I : \mathfrak{o}(f,x) \geq \varepsilon\}$ is closed (in particular, B_ε is compact).

   Proof. We will prove that $B_\varepsilon^c \cap I$ is relatively open in I. Let $x \in I$ with $\mathfrak{o}(f,x) < \varepsilon$. Let $\delta > 0$ be such that $M(x,f,\delta) - m(x,f,\delta) < \varepsilon$. Let $y \in \mathcal{B}_\delta(x)$ and take $\delta_y > 0$ such that $\mathcal{B}_{\delta_y}(y) \subset \mathcal{B}_\delta(x)$, then

   $$M(y,f,\delta_y) - m(y,f,\delta_y) \leq M(x,f,\delta) - m(x,f,\delta) < \varepsilon$$

   giving that $\mathfrak{o}(f,y) < \varepsilon$. Thus B_ε is relatively closed in I, so B_ε is closed. □

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   Notice that $B_\varepsilon \subset B_f$ by claim 1, hence B_ε has measure zero. Thus B_ε has content zero by Theorem (14.3). Let $\varepsilon > 0$ be given, let $U_1, U_2, \ldots, U_n$ be rectangles such that $B_\varepsilon \subset \bigcup_{j=1}^n U_j^\circ$ (union of intervals) and $\sum_{j=1}^n \mu(I_j) < \varepsilon$. Let P'_ε be a partition of I such that for each $\alpha \in P'_\varepsilon$, the rangles I_α has one of the following properties:

   (a) $I_\alpha \subset U_j$ for some $j = 1, 2, \ldots, n$, or;

   (b) $I_\alpha \cap B_\varepsilon \neq \varnothing$.

This can be done by considering the rectangles $U_j \cap I$, and because if

$$I_\alpha \cap \left[\bigcup_{j=1}^{n} (U_j \cap I)^\circ\right] = \varnothing$$

then $I_\alpha \cap B_\varepsilon = \varnothing$. Let $M \geq 0$ be such that $|f(x)| \leq M$ for all $x \in I$, then

$$|f(x_\alpha) - f(y_\alpha) \leq 2M \qquad \forall\, x_\alpha, y_\alpha \in I_\alpha|$$

Now we get

$$\begin{aligned}\left|\sum_{I_\alpha \subset U_j \text{ for some j}} [f(x_\alpha) - f(y_\alpha)]\, \mu(I_\alpha)\right| &\leq \sum_{I_\alpha \subset U_j \text{ for some j}} |f(x_\alpha) - f(y_\alpha)|\, \mu(I_\alpha) \\ &\leq 2M \sum_{I_\alpha \subset U_j \text{ for some j}} \mu(I_\alpha) \\ &\leq 2M \sum_{j=1}^{n} \mu(U_j) = 2M\varepsilon\end{aligned}$$

(a) *Claim 3*: If $\alpha \in P'_\varepsilon$ and $I_\alpha \cap B_\varepsilon = \varnothing$, then there exists a partition P_α of I_α such that

$$|f(x_\beta) - f(y_\beta)| \leq 2\varepsilon \qquad \forall\, x_\beta, y_\beta \in J_{\alpha,\beta}$$

where $J_{\alpha,\beta}$ is a subrectangle in the in the subdivision corresponding to P_α.

Proof. Since $I_\alpha \cap B_\varepsilon = \varnothing$, we have $ø(f, x) < \varepsilon$ for all $x \in I_\alpha$. For each $x \in I_\alpha$, let $\delta_x > 0$ be such that

$$|f(y) - f(z)| < \varepsilon \qquad \forall\, y, z \in \mathcal{B}_{\delta_x}(x)$$

then

$$I_\alpha \subset \bigcup_{x \in I_\alpha} \mathcal{B}_{\delta_x/2}(x)$$

Let $\{x_1, x_2, \ldots, x_\ell\}$ be such that

$$I_\alpha \subset \bigcup_{i=1}^{\ell} \mathcal{B}_{\delta_{x_i}/2}(x_i)$$

Take $\delta = \min\{\delta_{x_i}/2 : i = 1, \ldots, \ell\}$. Let P_α be a partition of I_α such that x, y belong to the subrectangles, we have $\|x - y\| < \delta$. It follows that if $x_\beta, y_\beta \in I_{\alpha,\beta}$, then taking i such that $x_\beta \in \mathcal{B}_{\delta_{x_i}/2}(x_i)$, we have $y_\beta \in \mathcal{B}_{\delta_{x_i}}(x_i)$. This gives $|f(x_\beta) - f(y_\beta)| < 2\varepsilon$. □

It follows by Claim 3 that we can find a refinement P_ε of P'_ε with the properties above and also with the additional property that $|f(x_\alpha) - f(y_\alpha)| < 2\varepsilon$, where $\alpha \in P_\varepsilon$ and $I_\alpha \cap B_\varepsilon = \varnothing$. Let $S_1(f, P_\varepsilon)$ and

$S_2(f, P_\varepsilon)$ be Riemann sums corresponding to P_ε, then

$$\begin{aligned}\left|\sum_{\alpha\in P_\varepsilon}[f(x_\alpha)-f(y_\alpha)]\,\mu(I_\alpha)\right| &\le \sum_{\alpha\in P_\varepsilon, I_\alpha\subset U_j}|f(x_\alpha)-f(y_\alpha)|\,\mu(I_\alpha)+\sum_{\alpha\in P_\varepsilon, I_\alpha\cap U_j=\varepsilon}|f(x_\alpha)-f(y_\alpha)|\,\mu(I_\alpha)\\ &\le 2M\varepsilon+2\varepsilon\mu(I)\end{aligned}$$

Thus by Corollary 14.1, f is Riemann Integrable.

2. ($\Longrightarrow$)
Suppose f is Riemann integrable, for each n, let

$$B_{1/n}=\{x\in I:\o(f,x)\ge\frac{1}{n}\}$$

By Claim 1,

$$B_f=\bigcup_{n=1}^{\infty}B_{1/n}$$

Thus STP that each $B_{1/n}$ has measure zero (in fact, content zero). Fix n and let $\varepsilon>0$. Let P_ε be a partition of I such that

$$S_1(f,P_\varepsilon)-S_2(f,P_\varepsilon)<\frac{\varepsilon}{n}$$

for all Riemann sums $S_1(f, P_\varepsilon)$ and $S_2(f, P_\varepsilon)$. Write

$$B_{1/n}=C_1\cup C_2 \qquad \text{where } C_1=\{x\in B_{1/n}:x\in\partial I_\alpha \text{ for some } \alpha\}$$
$$C_2=\{x\in B_{1/n}:x\in I_\alpha^\circ \text{ for some } \alpha\}$$

Then C_1 has content zero because each I_α does. Let

$$\mathbb{S}=\{I_\alpha:I_\alpha^\circ\cap C_2\ne\varnothing\}$$

Then $C_2\subset\bigcup_{I_\alpha\in\mathbb{S}}I_\alpha$. Given $\varepsilon'>0$, $\varepsilon'<1/n$, for each $I_\alpha\in\mathbb{S}$, we can find $x_\alpha, y_\alpha\in I_\alpha$ such that

$$f(x_\alpha)-f(y_\alpha)>\frac{1}{n}-\varepsilon'$$

since $I_\alpha^\circ\cap C_2\ne\varnothing$. It follows that

$$\begin{aligned}0\le\sum_{I_\alpha\in\mathbb{S}}\left(\frac{1}{n}-\varepsilon'\right)\mu(I_\alpha)&\le\sum_{I_\alpha\in\mathbb{S}}\left(f(x_\alpha)-f(y_\alpha)\right)\mu(I_\alpha)\\ &=S_1(f,P_\varepsilon)-S_2(f,P_\varepsilon)<\frac{\varepsilon}{n}\end{aligned}$$

Since $\varepsilon'>0$, this yields that

$$\sum_{I_\alpha\in\mathbb{S}}\frac{\mu(I_\alpha)}{2}\le\frac{\varepsilon}{2}\ \Rightarrow\ \sum_{I_\alpha\in\mathbb{S}}\mu(I_\alpha)\le\varepsilon$$

so C_2 has content zero as needed.

□

Corollary 14.2

Let $\varnothing \neq D \subset \mathbb{R}^N$ be bounded, TFAE:

1. D has content;
2. ∂D has content zero.

Corollary 14.3

Let $\varnothing \neq D \subset \mathbb{R}^N$ be bounded and ∂D has content zero. If $f : D \to \mathbb{R}^M$ is continuous, then f is Riemann integrable.

Corollary 14.4

Let $f : I \to \mathbb{R}^M$ and suppose the set of points at which f is discontinuous is countable, then f is Riemann integrable.

Proposition 14.4: Properties of the Riemann integrable

Let $\varnothing \neq D \subset \mathbb{R}^N$ be bounded, let $f, g; D \to \mathbb{R}^M$ be Riemann integrable, then

1. $f + g$ is Riemann integrable, and;
$$\int (f+g) = \int f + \int g$$
2. $\|f\| : D \to \mathbb{R}$, $x \mapsto \|f(x)\|$ is Riemann integrable, and;
3. If $M = 1$, $f \leq g$, then
$$\int f \leq \int g$$
4. If $M = 1$, D has content and $r \leq f \leq R$, then
$$r\mu(D) \leq \int f \leq R\mu(D)$$

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14.5.1 Mean Value Theorem for Integration

Theorem 14.5: Mean Value Theorem for Integration

Let $\varnothing \neq D \subset \mathbb{R}^N$ and $f : D \to \mathbb{R}$ continuous on D. Suppose that D is compact, connected and has content. Then there exists $x_0 \in D$ such that

$$\int_D f = f(x_0)\mu(D)$$

Proof. Since D has content, and f is continuous, then f is Riemann integrable by Corollary (14.3). Let $r, R \in \mathbb{R}$ such that

$$r \leq f \leq R$$

By extreme value theorem, there are $p, q \in D$ such that

$$f(p) = r \qquad \text{and} \qquad f(q) = R$$

We have

$$r\mu(D) \leq \int f \leq R\mu(D)$$

so if $\mu(D) = 0$, $\displaystyle\int f = 0$ and any $x_0 \in D$ satisfies the result. Assume $\mu(D) \neq 0$ and let

$$\lambda := \frac{\int f}{\mu(D)}$$

so $f(p) \leq \lambda \leq f(q)$ and since D is connected, there exists by the intermediate value theorem $x_0 \in D$ such that

$$f(x_0) = \lambda = \frac{\int f}{\mu(D)}$$

□

14.6 Fubini's Theorem

How do we actually calculate the integral, $\displaystyle\int_D f$?

Example 14.4

Using a simple exmaple to show the idea: Suppose $I = [a, b] \times [c, d] \subset \mathbb{R}^2$ and $f : I \to \mathbb{R}$ continuous, $f \geq 0$. Hence $\displaystyle\int f$ is the volume of the region under the graph of f. In particular, we have

$$\int_I f = \int_a^b \left(\int_c^d f(x, y)\ dy \right)\ dx = \int_c^d \left(\int_a^b f(x, y)\ dx \right)\ dy$$

It could happen in general that for some x, the function $y \mapsto f(x, y)$ is not Riemann Integrable.

Theorem 14.6: Fubini's Theorem

Let $I \subset \mathbb{R}^N$ and $J \subset \mathbb{R}^M$ be rectangles and $f : I \times J \to \mathbb{R}^K$ be Riemann Integrable. Suppose that for each $x \in I$ the function $y \in J \mapsto f(x,y) \in \mathbb{R}^K$ is Riemann integrable and let

$$h(x) = \int_J f(x,y)\, dy \qquad (x \in I)$$

Then h is integrable and

$$\int_I \left(\int_J f(x,y)\, dy \right)\, dx = \int_I h(x)\, dx = \int_{I\times J} f$$

Discovery 14.7

A similar statement holds if $x \mapsto f(x,y)$ is integrable for each $y \in J$ and we let $g(y) = \int_I f(x,y)\, dx$.

Proof. We may assume that K is 1 by A5Q2. Let $\varepsilon > 0$ be given and P_ε be a partition of $I \times J$ such that

$$\left| S(f,P) - \int f \right| < \frac{\varepsilon}{2}$$

for all refinement P of P_ε and all Riemann sum corresponding to P. Let P_ε^I and P_ε^J be partitions of I and J respectively, so that

$$P_\varepsilon = P_\varepsilon^I \times P_\varepsilon^J$$

Let P^I and P^J be refinements of P_ε^I and P_ε^J respectively and for each $\alpha \in P^I$ and $\beta \in P^J$ chosoe $x_\alpha \in I_\alpha$ and $y_\beta \in J_\beta$, then the above inequality yields

$$\left| \sum_{(\alpha,\beta)\in P^I\times P^J} f(x_\alpha, y_\beta)\mu(I_\alpha \times J_\beta) - \int_{I\times J} f \right| < \frac{\varepsilon}{2}$$

Then since $\mu(I_\alpha \times J_\beta) = \mu(I_\alpha)\mu(J_\beta)$, we get

$$\left| \sum_{\alpha\in P^I} \left(\sum_{\beta\in P^J} f(x_\alpha, y_\beta)\mu(J_\beta) \right) \mu(I_\alpha) - \int_{I\times J} f \right| < \frac{\varepsilon}{2}$$

Fix P^I and $x_\alpha \in I_\alpha$, let Q_ε^J be a refinement of P_ε^J such that

$$\left| \sum_{\beta\in Q_\varepsilon^J} f(x_\alpha, y_\beta)\mu(J_\beta) - h(x_\alpha) \right| < \frac{\varepsilon}{2\mu(I)}$$

for all $\alpha \in P_I$. Then combining this with the previous inequality, we have

$$\begin{aligned} &\left| \sum_{\alpha \in P^I} \left(\sum_{\beta \in Q_\varepsilon^J} f(x_\alpha, y_\beta)\mu(J_\beta) \right) \mu(I_\alpha) - \sum_{\alpha \in P_J} h(x_\alpha)\mu(I_\alpha) \right| \\ &\le \sum_{\alpha \in P^I} \left| \sum_{\beta \in Q_\varepsilon^I} f(x_\alpha, y_\beta)\mu(I_\beta)\mu(I_\alpha) - h(x_\alpha)\mu(I_\alpha) \right| \\ &< \sum_{\alpha \in P_I} \frac{\varepsilon}{2\mu(I)} \cdot \mu(I) = \frac{\varepsilon}{2} \end{aligned}$$

Thus we know that

$$\left| \sum_{\alpha \in P^I} h(x_\alpha)\mu(I_\alpha) - \int_{I \times J} f \right| < \varepsilon$$

which implies that h is integrable and $\displaystyle\int h(x)\, dx = \int f$. $\square$

Corollary 14.5

Let $I = [a, b] \times [c, d] \subset \mathbb{R}^2$ and $f : I \to \mathbb{R}$ be integrable. Suppose that the function

$$y \mapsto f(x, y) \qquad \text{and} \qquad x \mapsto f(x, y)$$

are integrable for all $x \in [a, b]$ and $y \in [c, d]$, then

$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_I f = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Example 14.5

Let $I = [0, 1] \times [0, 1]$ and let $f(x, y) = y^3 e^{xy^2}$. Then

$$\begin{aligned} \int_0^1 \left(\int_0^1 y^3 e^{xy^2}\, dy \right) dx &= \int_0^1 \left(\int_0^1 y^3 e^{xy^2}\, dx \right) dy \\ &= \int_0^1 \frac{y^3 e^{xy^2}}{y^2} \bigg|_0^1 dy \\ &= \int_0^1 y \left(e^{y^2} - 1 \right) dy = \frac{e}{2} - 1 \end{aligned}$$

Corollary 14.6

Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be continuous and let $D = \{(x, y) : x \in [a, b],\ \text{and}\ \varphi(x) \le y \le \psi(x)\} \subset \mathbb{R}^2$.

Suppose that $F : D \to \mathbb{R}$ is continuous, then

$$\int_D f = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\, dy \right) dx$$

Proof. Notice that ∂D has content zero because it is the finite union of graphs of continuous functions on compact set. By Corollary (14.3) f is integrable. Let $I = [a,b] \times [c,d]$ containing D and $\tilde{f}$ the extension of f to I by $\tilde{f}(x) = 0$ for $x \notin D$. For $x \in [a,b]$ fixed, the function $y \mapsto \tilde{f}(x,y)$ is continuous on $[c,d]$ at $\varphi(x)$ and $\psi(x)$. By Fubini

$$\begin{aligned}\int_D f = \int_I \tilde{f} &= \int_a^b \int_c^d \tilde{f}(x,y)\, dy\, dx \\ &= \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\, dy\, dx\end{aligned}$$

as desired. □

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Example 14.6

Let $D = \{(x,y) : 1 \le x \le 3, x^2 \le y \le x^2 + 1\}$. Compute the content (the area) of D.

Proof. We have by the above Corollary that

$$\int_D 1 = \int_1^3 \int_{x^2}^{x^2+1} 1\, dy\, dx = 2$$

□

Example 14.7

Compute $\int_D f$ where $f(x,y,z) = y$ and D is the region bounded by the plane $z = 0$, $x = 0$, $y = 0$ and $x + y + z = 1$.

Proof. We can desciribe D as following:

$$0 \le x \le 1 \quad 0 \le y \le 1 - x \quad 0 \le z \le 1 - x - y$$

Thus by Fubini's Theorem and the above Corollary we have

$$\int_D f = \int_{[0,1]^3} \tilde{f} = \int_{[0,1]} \int_{[0,1]^2} \tilde{f} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y\, dz\, dy\, dx = \frac{1}{24}$$

Note: other ways to describe D could be, for example

$$0 \le z \le 1 \quad 0 \le x \le 1 - z \quad 0 \le y \le 1 - z - x$$

□

14.7 Change of Variables

Consider the function $f(x,y) = \dfrac{1}{(x^2+y^2)^{3/2}}$ defined on D where $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$. We wish to compute $\int_D f$.

Discovery 14.8

The idea is to use polar coordinates.

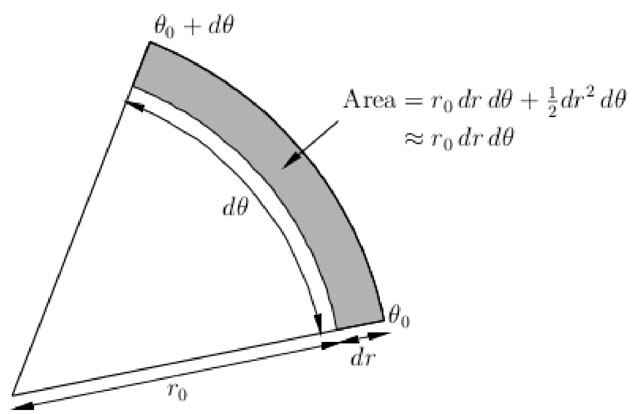
Suppose we have $g(r,\theta) = (r\cos\theta, r\sin\theta)$, then

$$D = g(A) \quad \text{where } A = \{(r,\theta) : 1 \leq r \leq 2, 0 \leq \theta < 2\pi\}$$

Hence D is replaced by a rectangle.
Also

$$(f \circ g)(r,\theta) = \frac{1}{r^3}$$

so everything looks simple. Can we compute $\int_D f$ in terms of $f \circ g$?
Consider an infinitesimal pizza-like box in polar coordinate:



The area of the shaded region would be

$$\frac{r^2\, d\theta}{2} - \frac{(r^2 - dr)\, d\theta}{2} \approx r\, dr\, d\theta \quad \text{if } dr \approx 0$$

so

$$\int_D f = \int_A f \circ g\, dA = \int_A f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

Theorem 14.7: Change of Variable Theorem

Let $\varnothing \neq U \subset \mathbb{R}^N$ be open and let $\varnothing \neq K \subset U$ be compact with content. Suppose $g : U \to \mathbb{R}^N$ is continuously differentiable and suppose that there exists $Z \subset K$ with content zero such that

1. g is one-to-one on $K \backslash Z$;

2. $\det J_g(x) \neq 0$ for all $x \in K \backslash Z$,

then $g(K)$ has content and for every $f : g(K) \to \mathbb{R}$ continuous we have

$$\int_{g(K)} f = \int_K (f \circ g)\,|\det J_g|$$

where $\det J_g : K \to \mathbb{R}$ is defined as $x \mapsto \det J_g(x)$.

Example 14.8

Back to the example we had at the start. Consider $g(r,\theta) = (r\cos\theta, r\sin\theta)$, then $g \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. We have

$$J_g(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \quad \Rightarrow \quad \det J_g(r,\theta) = r$$

Notice that if $A = [1,2] \times [0, 2\pi]$, then $\det J_g(r,\theta) \neq 0$ on A and g is injective on $[1,2] \times [0, 2\pi)$. Since $[1,2] \times \{2\pi\}$ has content zero, we apply the Change of Variable Theorem:

$$\int_D f = \int_A f(r\cos\theta, r\sin\theta) r\; dr\; d\theta = \int_1^2 \int_0^{2\pi} \frac{1}{r^2}\; d\theta\; dr = \int_1^2 \frac{2\pi}{r^2}\; dr = \pi$$

14.8 Integration with Cylindrical Coordinates

The sylindrical coordinates in $\mathbb{R}^3$ are

$$x = r\cos\theta \quad y = r\sin\theta \quad z = z$$

Thus

$$g : \mathbb{R}^3 \to \mathbb{R}^3 \qquad g(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$$

Then g is continuously differentiable and

$$J_g(r,\theta,z) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $\det J_g(r,\theta,z) = r$.

Example 14.9

Find the volume of the region D in $\mathbb{R}^3$ above the paraboloid $z = x^2 + y^2$, and inside the sphere $x^2 + y^2 + z^2 = 12$.

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Proof. Write that $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$. On the paraboloid, we have $z = r^2$ while on the sphere, we have $r = \sqrt{12 - r^2}$. We now want to find the the value of r where the paraboloid and the sphere meets:

we have

$$r_{\max}^2 + r_{\max}^4 = 12 \quad \Rightarrow \quad r_{\max} = \sqrt{3}$$

Hence $D = g(K)$ where $g(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ and

$$K = \{(r, \theta, z) : 0 \le r \le \sqrt{3}, 0 \le \theta \le 2\pi, r^2 \le z \le \sqrt{12 - r^2}\}$$

By the Change of Variable Theorem,

$$\mu(D) = \int_D 1 = \int_K r\, dz\, d\theta\, dr = \int_0^{\sqrt{3}} \int_0^{2\pi} \int_{r^2}^{\sqrt{12-r^2}} r\, dz\, d\theta\, dr = 2\pi\left[-\frac{45}{4} + \frac{12^{3/2}}{3}\right]$$

□

14.9 Spherical Coordinates

In the system of spherical coordinates, we have the following coordinate axes:

1. ρ: the distance to the origin, so that $x^2 + y^2 + z^2 = \rho^2$, $(\rho \ge 0)$;
2. θ: "longitude" angle from the positive x-axis, $(0 \le \theta \le 2\pi)$;
3. φ: "latitude" angle from the positive z-axis, $(0 \le \varphi \le \pi)$.

Definition 14.9

We wish to denote (x, y, z) in terms of (ρ, θ, φ):

$$z = \rho\cos\varphi \qquad x = \rho\sin\varphi\cos\theta \qquad y = \rho\sin\varphi\sin\theta$$

Discovery 14.9

Consider $g : \mathbb{R}^3 \to \mathbb{R}^3$ where

$$g(\rho, \theta, \varphi) = (\rho\cos\varphi, \rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta)$$

so $g \in C1(\mathbb{R}^3, \mathbb{R}^3)$ and g is injective on

$$\{(\rho, \theta, \varphi) : \rho > 0, 0 \le \theta < 2\pi, 0 \le \varphi \le \pi\}$$

Moreover

$$J_g(\rho, \theta, \varphi) = \begin{bmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\rho & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{bmatrix}$$

so

$$\det J_g(\rho, \theta, \varphi) = -\rho^2\sin\varphi$$

Hence $\det J_g(\rho, \theta, \varphi) \neq 0$ if $\rho \neq 0$ and $\varphi \neq 0, \pi$.

Example 14.10: Example in Spherical Coordinates

Suppose $\rho = r$ is a non-zero constant and φ is also a constant not equal to 0 or π, then we get a cone with vertex at the origin.

Example 14.11

Compute the volume of the sphere with radius r using spherical coordinates:

Proof. We have

$$D = \{(x, y, z) : x^2 + y^2 + z^2 = r^2\}$$

hence $D = g(K)$, where $g(\rho, \theta, \varphi) = (\rho\cos\varphi, \rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta)$ and

$$K = \{(\rho, \theta, \varphi) : 0 < \rho < r, 0 \le \theta < 2\pi, 0 \le \varphi \le \pi\}$$

Then

$$\mu(D) = \int_D 1 = \int_K |\det J_g| = \int_0^r \int_0^\pi \int_0^{2\pi} \rho^2 \sin\varphi \, d\theta \, d\varphi \, dr = \frac{4\pi}{3} r^3$$

□

Result 14.1: Idea of the proof for the Change of Variable Theorem

Suppose $I = [a_1, b_1] \times \cdots \times [a_N, b_N]$ and $a = (a_1, \ldots, a_N)$, then

$$I = \{a_1 + h_1 e_1 + \cdots a_N + h_N e_N : 0 \le h_k \le \ell_k \text{ for } 0 \le k \le N\}$$

where $\ell_k = b_k - a_k$. If I is very small,

$$g(I) \approx \left\{ g(a) + D_g(a) \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}, 0 \le h_k \le \ell_k \text{ for } k = 1, \ldots, N \right\}$$

where

$$D_g(a) = \begin{bmatrix} | & & | \\ D_g(a)e_1 & \cdots & D_g(a)e_N \\ | & & | \end{bmatrix}$$

Then column vectors are linearly independent, and they form a parallelepiped. We observe that

$$\mu(\text{par}) = \left| \det \begin{bmatrix} | & & | \\ \ell_1 D_g(a)e_1 & \cdots & \ell_N D_g(a)e_N \\ | & & | \end{bmatrix} \right| = \mu(I) |\det J_g(a)|$$

Thus

$$\mu(g(I)) \approx \mu(I) |\det J_g(a)| \quad \Rightarrow \quad \int_I |\det J_g|$$

In general, take partition P of I,

$$\int g(K)I \approx \sum_{\alpha\in P}\int_{g(I_\alpha)} f = \sum_{\alpha\in P} f(y_\alpha)\int_{g(I_\alpha)} 1 = \sum_{\alpha\in P} f(y_\alpha)\mu(g(I_\alpha)) = \sum_{\alpha\in P} f(y_\alpha)\int_{I_\alpha} \det J_g$$

which yields $\int_I f \circ g|\det J_g|$.

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